Efficient Risk Estimation for Orbital Collisions Using Hierarchical Monte Carlo Methods

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Introduction & Outline

- Abdul-Lateef (Al) Haji-Ali, Associate Professor, Heriot-Watt University and Humboldt Foundation Research Fellow (currently visiting RWTH Aachen).
- Research in Uncertainty Quantification in
  - (adaptive) sampling and Monte Carlo-based methods,
  - numerical methods for SDEs and PDEs (Stochastic and Partial Differential Equations),
  - particle systems,
  - and risk assessment.

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  - numerical methods for SDEs and PDEs (Stochastic and Partial Differential Equations),
  - particle systems,
  - and risk assessment.
- **This talk**:
  - Objective: Present my research in the context of Collision Detection.
  - Goal: Convince you that "gold standard" methods are tenable.
  - Aim: Explore collaboration opportunities.
- **Outline**
  - Present my understanding of the domain problem.
  - Discuss my preferred methodology and its computational problems.
  - Talk about solutions.

Motivation

- Over $\sim$43,500 catalogued objects currently orbiting the Earth (only $\sim$13,000 are operational spacecrafts).
- Need to track collision events of all objects (and the increased space debris) for assessing risk in missions and satellite deployment.
- Need to track generation of new space debris due to collisions and subsequent effect on collision probability.
- The setup is simple (mathematically)

$$\mathrm{P}\left[\inf\left\{t : \min_{i\neq j}(|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)| - w_{ij}) \leq 0\right\} \leq T\right]$$

  where $\boldsymbol{r}_i$ is the position of object i.
- Similar goal when tracking: compute the probability that an observation comes from the same object.

Techniques in Orbital Collision Detection

Typical assumptions during encounter:

- Gaussian distributions of objects' state at conjunction time.
- Rectilinear motion is assumed due to the high relative velocities involved.
- Static velocity covariance, i.e., uncertainty in velocity does not change during encounter.
- Spacecraft are modelled as spherical objects, i.e., simplified geometry for collision modelling.

Non-linearity and non-Gaussianity

These assumptions are problematic when velocities are low or when non-Gaussianity is strong (especially during long-term encounters). Other methods try to address this:

- Numerical integration (Foster and Estes 1992; Akella and Alfriend 2000) – integration can be smoothed with analytical techniques in direction (Patera 2003).
- Series expansion to address non-linearity with Gaussian assumptions (Alfano 2005; Chan 1997; García-Pelayo and Hernando-Ayuso 2016).
- Polynomial chaos expansion (Jones and Doostan 2013) and Gaussian mixture models (DeMars, Cheng, and Jah 2014; Vittaldev and Russell 2013; Shelton and Junkins 2019) (and others) to deal with non-gaussianity.

Accurate propagation first

Orbit Propagation remains an issue:

- Keplerian models, $\gtrsim 10\mathrm{km}$ error after 1 day.
- SGP/SDP models, $\leq 10\mathrm{km}$ error after 1 day.
- HF integrators with full (potentially stochastic) force models, ~ 10 meters error after 1 day.
- Surrogate models deal with this, but require expensive training and error analysis is sometimes difficult.

Considering many objects, especially with uncertain positions and velocities, is also expensive.

Methodology: Monte Carlo Simulation

The gold standard method – Monte Carlo simulation of all orbits with HF integrators – is far too expensive.

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Main sources of **compounding** cost:

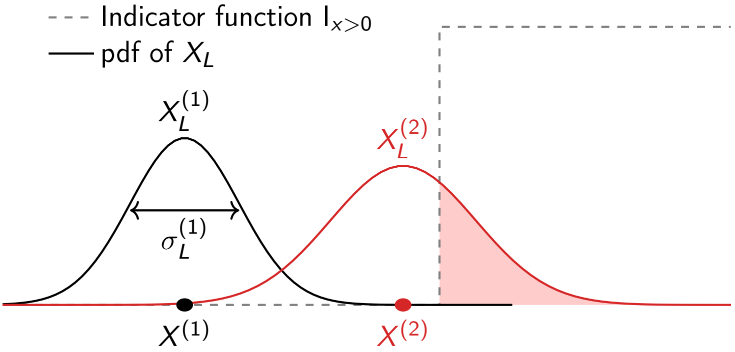
- HF propagation $\rightarrow$ Adaptive and Hierarchical sampling.
- Rarity of event $\rightarrow$ Importance sampling.
- Cost of sampling $\rightarrow$ HPC and Quasi Monte-Carlo.
- Number of objects (especially from a cloud) $\rightarrow$ Hierarchical sampling.

HF Propagation: Adaptive Sampling

Assume we want to compute

$$\mathrm{P}[\,X > 0\,] \approx \mathrm{P}[\,X_L > 0\,]$$

where $X_L \approx X$ with increasing accuracy and cost (think different models or different resolutions).



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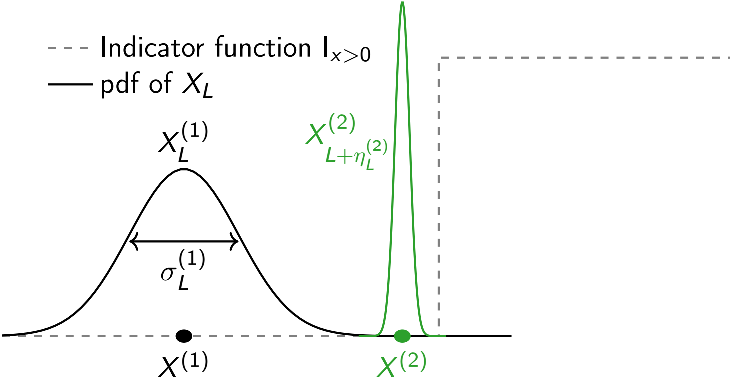

HF Propagation: Hierarchical Sampling

Further reduction in cost by considering a hierarchy of approximations $(X_\ell)_{\ell \geq 0}$

$$
\begin{aligned}
\mathrm{P}[\, X > 0\,] &\approx \mathrm{P}[\, X_L \quad > 0\,] \\
&= \mathrm{P}[\, X_0 \; > 0\,] + \sum_{\ell=1}^{L} \mathrm{P}[\, X_\ell \quad > 0\,] - \mathrm{P}[\, X_{\ell-1} \qquad > 0\,]
\end{aligned}
$$

We still approximate all probabilities using Monte Carlo simulations. However, the cost of computing the difference of probabilities is much smaller than computing one probability (assuming correlated X_ℓ and $X_{\ell-1}$).

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$$\begin{aligned} \mathrm{P}[\, X > 0 \,] &\approx \mathrm{P}[\, X_{L+\eta_L} > 0 \,] \\ &= \mathrm{P}[\, X_{\eta_0} > 0 \,] + \sum_{\ell=1}^{L} \mathrm{P}[\, X_{\ell+\eta_\ell} > 0 \,] - \mathrm{P}[\, X_{\ell-1+\eta_{\ell-1}} > 0 \,] \end{aligned}$$

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In some cases, one obtains a computational complexity which does not "see" the cost of the HF propagation (my previous work[1] showed this when, e.g., computing probabilities of large loss in financial applications).

[1] **H-A**, Spence, and Teckentrup 2022.

Rarity of event: Importance Sampling

- Number of samples to estimate a 10^{-q} probability to relative accuracy ε is $\sim \varepsilon^{-2} 10^{q}$.
- The standard way to deal with this is importance sampling.

$$\mathrm{P}_p[\, X > 0 \,] = \int_0^\infty p(x)\,\mathrm{d}x = \int_0^\infty \frac{p(x)}{q(x)} q(x)\,\mathrm{d}x = \mathrm{E}_q\left[\frac{p(X)}{q(X)} \right]$$

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- Finding a suitable q can be hard, but does not need to be accurate – techniques such as Adaptive Importance Sampling, Sequential Monte Carlo or Stochastic Optimal Control can be used.
- These techniques can use inaccurate propagation models, or again a hierarchy of such models.

See, e.g., my previous work[2] showing the use of IS in different settings.

[2] Ben Rached, **H-A**, Subbiah Pillai, et al. 2024; Ben Amar et al. 2023; Ben Rached, **H-A**, Rubino, et al. 2021.

Cost of sampling: HPC and Quasi-Monte Carlo

- Monte Carlo methods are highly parallelizable.
- High-performance compute is becoming easier to come by, but sometimes harder to fully exploit.
- Adaptive, importance sampling and hierarchical methods inherit this feature (with some care).
- Better sampling algorithms with fast convergence, such as Quasi-Monte Carlo, can be used as a “almost” drop-off replacement since they can be made unbiased.

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- Better sampling algorithms with fast convergence, such as Quasi-Monte Carlo, can be used as a “almost” drop-off replacement since they can be made unbiased.
    - Although direct application is not possible,
    - numerical or analytical smoothing of the discontinuity is simple and effective, e.g., I used smoothing[3] to compute probabilities in financial applications.

[3] Giles and **H-A** 2024.

Secondary collisions

Preparing for the space of tomorrow

- With increasing number of objects, collisions will become more "common".
- Predicting first-order collisions over a short time horizon is "easy".
- Predicting first-order collisions over a longer time horizons is still possible using HF propagation.
- Higher-order collisions will soon become critical to model and assess.
  - Position/velocity of new debris is uncertain.
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- Doubly nested simulation. Double the trouble?

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- Can be handled easily using simulation, when a collision happens, add a random number of objects to the catalogue and branch simulation.
- Doubly nested simulation. Double the trouble? Not necessarily – we were able to use hierarchical and adaptive nested simulation with barely any increased computational complexity![4].

[4] Giles and **H-A** 2019.

Conclusion

(Quasi) Monte Carlo coupled with advanced (yet simple) techniques:

- adaptivity,
- hierarchical sampling,
- importance sampling,
- and HPC

makes full-fidelity assessment of collision risk feasible (even for large-scale and long-term horizon).

Thank you. Questions?

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