A multi-index Monte Carlo method for semilinear parabolic SPDEs

Abdul-Lateef Haji-Ali.
In collaboration with Håkon Hoel (University of Oslo) and Andreas Petersson (Linnaeus University).

Heriot-Watt University

EPFL, Switzerland

Problem description

We consider the H-valued SDE

$$\mathrm{d}X(t) + \big(AX(t) - F(X(t))\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t), \quad t \in (0, T]$$
$$X(0) = X_0 \in H$$

Goal: Approximate $\mathbb{E}[\Psi(X(T))]$ for smooth QoI $\Psi : H \to U$.

Contribution: A multi-index Monte Carlo method

$$\mu_{\mathrm{MI}} := \sum_{\boldsymbol{\ell} \in \mathcal{I} \subset \mathbb{N}_0^2} \sum_{i=1}^{m_{\boldsymbol{\ell}}} \frac{\Psi(X_{N_{\ell_2}}^{M_{\ell_1},\boldsymbol{\ell},i}) - \Psi(X_{N_{\ell_2}}^{M_{\ell_1-1},\boldsymbol{\ell},i}) - \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1},\boldsymbol{\ell},i}) + \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1-1},\boldsymbol{\ell},i})}{m_{\boldsymbol{\ell}}}$$

that achieves

$$\mathbb{E}[\|\mu_{\mathrm{MI}} - \mathbb{E}[\Psi(X(T))]\|_U^2] \lesssim \varepsilon^2$$

at a computational cost of almost $\mathcal{O}(\varepsilon^{-2})$ [1].

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[1] under favourable conditions.

Overview

1. The SPDE and mild solutions
2. Monte Carlo methods for approximating $\mathbb{E}[\Psi(X(T))]$
3. Accelerated exponential integrator method
4. Numerical experiments and conclusion

Canonical example: Stochastic heat equation

$$\mathrm{d}X(t) + \big(-\Delta X(t) - F(X(t))\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t), \qquad \text{in } [0,T]\times\mathcal{D}$$
$$X(t,\cdot)|_{\partial\mathcal{D}} = 0, \qquad X(0,\cdot) = X_0\,.$$

- Here $X \in L^2(\mathcal{D})$ for a bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $d = 1,2,3$, convex or with $\mathcal{C}^2$ boundary $\partial\mathcal{D}$.
- $-\Delta : \mathrm{Dom}(-\Delta) = W^{2,2}(\mathcal{D}) \subset L^2(\mathcal{D}) \to L^2(\mathcal{D})$ is a densely defined, positive definite linear operator with orthonormal eigenbasis

$$\big((e_j,\lambda_j)\big)_{j=1}^{\infty}, \qquad \text{where } \lambda_j \eqsim j^{2/d}, \qquad \text{by Weyl's law.}$$

- We consider $\mathcal{D}$ for which (e_j,λ_j) are known. E.g., $\mathcal{D} = (0,1)$ with

$$e_j(x) = \sqrt{2}\sin(j\pi x), \qquad \text{and} \quad \lambda_j = \pi^2 j^2.$$

- A natural example of F is a composition (Nemytskii) mapping:

$$(F(u))(x) = f(u(x))$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is sufficiently smooth with bounded derivatives.

The General SPDE

We consider

$$\mathrm{d}X(t) + \big(AX(t) - F(X(t))\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t)$$

on Hilbert space $H = (H, \langle \cdot, \cdot \rangle, \| \cdot \|)$ and where: $A : \mathrm{Dom}(A) \subset H \to H$ is a

densely defined, positive definite linear operator with orthonormal eigenbasis

$$\big((e_j, \lambda_j)\big)_{j=1}^{\infty}, \qquad \text{where } \lambda_j \eqsim j^{\nu} \quad \text{for some } \nu > 0$$

Fractional operators $\quad A^r v := \sum_{j=1}^{\infty} \lambda_j^r \langle e_j, v \rangle e_j \quad \text{for} \quad r \in \mathbb{R}$

Extension of norm: $\| \cdot \|_{\dot{H}^r} := \|A^{r/2} \cdot \|$ and associated Hilbert space

$$\dot{H}^r = \begin{cases} \mathrm{Dom}(A^{r/2}) & r \geq 0 \\ \overline{H}^{\| \cdot \|_{\dot{H}^r}} & r < 0, \end{cases}$$

Q-Wiener process on Hilbert Space

W is a $Q-$Wiener process, meaning it has independent increments

$$W(t+\Delta t) - W(t) \sim \mathcal{N}(0, \Delta t Q).$$

Here, for any $\phi, \psi \in H, t \geq 0$,

$$\mathbb{E}[\langle W(t), \phi \rangle \langle W(t), \psi \rangle] = t \langle Q\phi, \psi \rangle,$$

for $Q \in \mathcal{L}_1^+(H)$ being a covariance operator (non-negative, trace-class, self-adjoint) with eigenpairs (e_k, μ_k) (same eigenbasis as operator A!).

Representation: $$W(t) = \sum_{j=1}^{\infty} \sqrt{\mu_j} B_j(t) e_j$$

with $B_j(t)$ independent, scalar-valued Brownian motions, defined on a complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$.

The noise operator $G \in \mathcal{L}\big(H, \mathcal{L}_2(Q^{1/2}(H), H)\big)$ is specified later.

Q-Wiener process

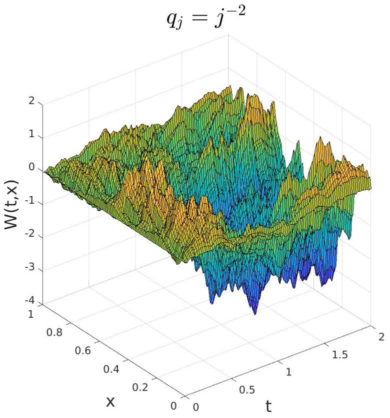


Figure 1: Regularity of W depends on decay rate of $(\mu_j = q_j)$

Mild solution[2]

If X_0 is $\mathcal{F}_0-$measurable and $X_0 \in L^p(\Omega, H)$, then there exists a unique mild solution in $C([0,T], L^p(\Omega, H))$ to the SPDE

$$\mathrm{d}X(t) + \big(AX(t) - F(X(t))\big)\,\mathrm{d}t = \big(I + GX(t)\big)\,\mathrm{d}W(t), \quad t \in (0,T]$$
$$X(0) = X_0.$$

Definition: a mild solution is an H-valued predictable process $\{X(t)\}_{t\in[0,T]}$ satisfying

$$X(t) = e^{-At}X_0 + \int_0^t e^{-A(t-s)}F(X(s))\,\mathrm{d}s + \int_0^t e^{-A(t-s)}\big(I + GX(s)\big)\,\mathrm{d}W(s)$$

for each $t \in [0,T]$. Here, $e^{-At}e_j = e^{-\lambda_j t}e_j$.

[2] Da Prato, G. & Zabczyk, J. *Stochastic equations in infinite dimensions.* Second, xviii+493 (Cambridge University Press, Cambridge, 2014).

Overview

Multilevel Monte Carlo (MLMC)[3]

Let X_N^M be a *general* numerical method with respect to integers M (e.g. timesteps) and N (e.g. eigenfunctions), and let $\Psi : H \to U$, for a real separable Hilbert space space $U = (U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$.
For integer sequences $(M_\ell)_{\ell=0}^\infty$ and $(N_\ell)_{\ell=0}^\infty$, consider the telescoping sum

$$\mathbb{E}[\Psi(X_{N_L}^{M_L}(T))] = \sum_{\ell=1}^{L} \mathbb{E}[\Psi(X_{N_\ell}^{M_\ell}(T)) - \Psi(X_{N_{\ell-1}}^{M_{\ell-1}}(T))] + \mathbb{E}[\Psi(X_{N_0}^{M_0}(T))].$$

This motivates the MLMC estimator based on sample averages E_{m_ℓ},

$$\mu_{\mathrm{ML}} := \sum_{\ell=1}^{L} E_{m_\ell}[\Psi(X_{N_\ell}^{M_\ell}(T)) - \Psi(X_{N_{\ell-1}}^{M_{\ell-1}}(T))] + E_{m_0}[\Psi(X_{N_0}^{M_0}(T))].$$

We assume that (here and below we ignore logarithmic terms!):

$$\mathrm{Cost}(\Psi(X_N^M)) \eqsim MN.$$

[3] Giles, M. B. Multilevel monte carlo path simulation. *Operations research* **56,** 607–617 (2008).

Multilevel Monte Carlo (MLMC)

Central point: given that

$$\|\Psi(X(T)) - \Psi(X_N^M(T))\|_{L^2(\Omega,H)}^2 \lesssim M^{-\beta_1} + N^{-\beta_2}$$

for some $\beta_1, \beta_2 \geq 0$, then with $M_\ell = 2^\ell$ and $N_\ell \eqsim 2^{\beta_1 \ell/\beta_2}$

$$\|\Psi(X_{N_\ell}^{M_\ell}(T)) - \Psi(X_{N_{\ell-1}}^{M_{\ell-1}}(T))\|_{L^2(\Omega,U)}^2 \lesssim 2^{-\ell\beta_1}$$

so very few samples m_ℓ needed when $\ell \gg 1$.
Performance[4]**:** For any $\varepsilon > 0$, there exists $(m_\ell)_\ell \subset \mathbb{N}$ s.t.

$$\mathbb{E}[\|\mu_{\mathrm{ML}} - \mathbb{E}[\Psi(X(T))]\|_U^2] \lesssim \varepsilon^2$$

with, given *weak convergence rates* α_1, α_2,
$\mathrm{Cost}(\mu_{\mathrm{ML}}) \lesssim \varepsilon^{-2-\max\left(0, \frac{1+\beta_1/\beta_2-\beta_1}{\min(\alpha_1, \alpha_2\beta_1/\beta_2)}\right)} \lesssim \varepsilon^{-2-2\max\left(0, \frac{1}{\beta_1}+\frac{1}{\beta_2}-1\right)}$

Monte Carlo cost is $\varepsilon^{-2-\frac{1}{\alpha_1}-\frac{1}{\alpha_2}} \leq \varepsilon^{-2-2\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)}$

[4] Chada, N. K., Hoel, H., Jasra, A. & Zouraris, G. E. Improved efficiency of multilevel Monte Carlo for stochastic PDE through strong pairwise coupling. *J. Sci. Comput.* **93,** Paper No. 62, 29 (2022).

Multi-index Monte Carlo (MIMC)[5]

Consider the telescoping double-sum

$$\mathbb{E}[\Psi(X_{N_L}^{M_L}(T))] = \sum_{\ell_1=0}^{L}\sum_{\ell_2=0}^{L}\mathbb{E}[\underbrace{\Psi(X_{N_{\ell_2}}^{M_{\ell_1}}) - \Psi(X_{N_{\ell_2}}^{M_{\ell_1-1}}) - \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1}}) + \Psi(X_{N_{\ell_2-1}}^{M_{\ell_1-1}})}_{=:\Delta_{\boldsymbol{\ell}}\Psi(X)}]$$

with $\Psi(X_{N_{\ell_2}}^{M_{\ell_1}}(T)) := 0$ whenever $\min(\ell_1,\ell_2) < 0$.
This motivates the MIMC estimator

$$\mu_{\mathrm{MI}} := \sum_{\boldsymbol{\ell}\in\mathcal{I}} E_{m_{\boldsymbol{\ell}}}[\Delta_{\boldsymbol{\ell}}\Psi(X)]$$

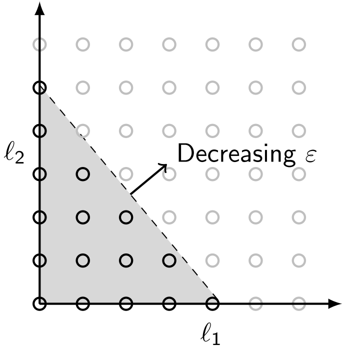
where $\mathcal{I} = \{\boldsymbol{\ell} = (\ell_1,\ell_2) \in \mathbb{N}_0^2 \mid \max(\ell_1,\ell_2) \le L\}$ and $(m_{\boldsymbol{\ell}})_{\boldsymbol{\ell}\in\mathbb{N}_0^2} \subset \mathbb{N}$.

[5] Haji-Ali, A.-L., Nobile, F. & Tempone, R. Multi-index Monte Carlo: when sparsity meets sampling. *Numerische Mathematik* **132,** 767–806 (2016).

The index set

Triangular index sets are more efficent than rectangular ones, so for some suitable weights (w_1, w_2) we actually use:

$$\mathcal{I} = \{\boldsymbol{\ell} \in \mathbb{N}_0^2 \mid w_1\ell_1 + w_2\ell_2 \leq L(\varepsilon)\}$$



MIMC for SPDE for general numerical method

Theorem

Suppose that $\Psi(X_N^M(T)) \to \Psi(X(T))$ *as* $M, N \to \infty$. *Assume also that we have* **a multiplicative** *bound on the* **second order difference**

$$\|\Delta_{\boldsymbol{\ell}} \Psi(X)\|^2_{L^2(\Omega,U)} \lesssim 2^{-\beta_1 \ell_1 - \beta_2 \ell_2}$$

for $\beta_1, \beta_2 > 0$. *Then, there is* $\alpha_i \geq \beta_i/2$ *such that*

$$\|\mathbb{E}[\Delta_{\boldsymbol{\ell}} \Psi(X)]\|_U \lesssim 2^{-\alpha_1 \ell_1 - \alpha_2 \ell_2}$$

and there exist MIMC parameters $\mathcal{I}$ *and* $(m_{\boldsymbol{\ell}}) \subset \mathbb{N}$ *s.t.*

$$\|\mu_{\mathsf{MI}} - \mathbb{E}[\Psi(X(T))]\|^2_{L^2(\Omega,U)} \lesssim \varepsilon^2$$

with

$$\mathrm{Cost}(\mu_{\mathsf{MI}}) \eqsim \varepsilon^{-2-2u} |\log \varepsilon^{-1}|^r$$

for some $0 < r \leq 4 + 2u$ *and*

$$2u = \max\left(0, \frac{1-\beta_1}{\alpha_1}, \frac{1-\beta_2}{\alpha_2}\right)$$

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Suppose that $\Psi(X_N^M(T)) \to \Psi(X(T))$ *as* $M, N \to \infty$. *Assume also that we have* **a multiplicative** *bound on the* **second order difference**

$$\|\Delta_{\boldsymbol{\ell}}\Psi(X)\|^2_{L^2(\Omega,U)} \lesssim 2^{-\beta_1\ell_1-\beta_2\ell_2}$$

for $\beta_1, \beta_2 > 0$. *Then, there is* $\alpha_i \geq \beta_i/2$ *such that*

$$\|\mathbb{E}[\Delta_{\boldsymbol{\ell}}\Psi(X)]\|_U \lesssim 2^{-\alpha_1\ell_1-\alpha_2\ell_2}$$

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$$\|\mu_{\mathsf{MI}} - \mathbb{E}[\Psi(X(T))]\|^2_{L^2(\Omega,U)} \lesssim \varepsilon^2$$

with

$$\mathrm{Cost}(\mu_{\mathsf{MI}}) \eqsim \varepsilon^{-2-2u}|\log \varepsilon^{-1}|^r$$

for some $0 < r \leq 4 + 2u$ *and*

$$2u = 2\max\left(0, \frac{1}{\beta_1} - 1, \frac{1}{\beta_2} - 1\right)$$

MIMC for SPDE for general numerical method

Theorem

Suppose that $\Psi(X_N^M(T)) \to \Psi(X(T))$ *as* $M, N \to \infty$. *Assume also that we have* **a multiplicative** *bound on the* **second order difference**

$$\|\Delta_{\boldsymbol{\ell}} \Psi(X)\|^2_{L^2(\Omega,U)} \lesssim 2^{-\beta_1 \ell_1 - \beta_2 \ell_2 - \vartheta \max(\ell_1 - \upsilon \ell_2, 0)}$$

for $\beta_1, \beta_2 > 0$. *Then, there is* $\alpha_i \geq \beta_i/2$ *such that*

$$\|\mathbb{E}[\Delta_{\boldsymbol{\ell}} \Psi(X)]\|_U \lesssim 2^{-\alpha_1 \ell_1 - \alpha_2 \ell_2 - (\vartheta/2) \max(\ell_1 - \upsilon \ell_2, 0)}$$

and there exist MIMC parameters $\mathcal{I}$ *and* $(m_{\boldsymbol{\ell}}) \subset \mathbb{N}$ *s.t.*

$$\|\mu_{\mathrm{MI}} - \mathbb{E}[\Psi(X(T))]\|^2_{L^2(\Omega,U)} \lesssim \varepsilon^2$$

with

$$\mathrm{Cost}(\mu_{\mathrm{MI}}) \eqsim \varepsilon^{-2-2u} |\log \varepsilon^{-1}|^r$$

for some $0 < r \leq 4 + 2u$ *and*

$$2u = 2 \max\left(0, \frac{1}{\beta_1 + \vartheta} - 1, \frac{1}{\beta_2} - 1, \frac{\upsilon + 1}{\upsilon \beta_1 + \beta_2} - 1\right)$$

MIMC schemes for SPDE[6]

- Applied to the Zakai equation

$$dX(t) + \left(\mu \frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) X(t)dt = -\sqrt{\rho}\left(\frac{\partial}{\partial x}X(t)\right) dB(t)$$

  with **scalar-valued** $B(t)$, constant $\rho \in [0,1)$ and $X(0) = \delta_{x_0}$ and $\mathcal{D} = \mathbb{R}$.
- Discretization by finite differences in space and Milstein in time leads to
  - Stability condition for explicit scheme: $(1+2\rho^2)\frac{\Delta t}{\Delta x^2} \leq 1 \implies$ **no good for MIMC**.
  - Implicit scheme is **unconditionally stable** for small ρ
- **MIMC for Zakai** with implicit scheme has complexity $\mathcal{O}(\varepsilon^{-2}|\log(\varepsilon)|^z)$ with $z \in \{1,3\}$, slightly **slower** than MLMC complexity $\mathcal{O}(\varepsilon^{-2})$.
- For our SPDE, we first tested implicit Euler MIMC, but did not see an improvement of MIMC over MLMC.
- Need to choose a suitable discretization scheme!

[6] Reisinger, C. & Wang, Z. Analysis of multi-index Monte Carlo estimators for a Zakai SPDE. English. *J. Comput. Math.* **36,** 202–236 (2018).

Overview

Accelerated exponential integrator method[7]

A *spectral numerical method* employing a semigroup shift.
Let $T > 0$, $M \in \mathbb{N}$ and $P_N : H \to \mathrm{Span}(e_1, \ldots, e_N) =: H_N$ be orthogonal projection.

$X_N^M : [0, T] \to H_N$ given by

$$X_N^M(t) = e^{-At} P_N X_0 + \int_0^t e^{-A(t-s)} P_N F(X_N^M(\lfloor s \rfloor_{M^{-1}})) \, \mathrm{d}s + \int_0^t e^{-A(t-s)} P_N (I + G X_N^M(\lfloor s \rfloor_{M^{-1}})) dW(s)$$

where $\lfloor t \rfloor_{M^{-1}} = \max\{ jT/M \; : \; jT/M < t, j = 0, 1, 2, \ldots, M \}$.

Note: When Q and A share an eigenbasis and when F is nonlinear, iterations can often be solved using FFT at an additional log-cost.

[7] Jentzen, A. & Kloeden, P. E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. English. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* **465,** 649–667 (2009).

Assumptions on X_0, Q, G

For a *regularity parameter* $\kappa \in (0,2)$:

1. $X_0 \in L^{10}(\Omega, \dot{H}^\kappa)$ is $\mathcal{F}_0$-measurable,
2. $Q^{1/2}(H) \overset{\mathcal{L}_2}{\hookrightarrow} \dot{H}^{\kappa-1}$, i.e., $\|A^{\frac{\kappa-1}{2}} Q^{\frac{1}{2}}\|_{\mathcal{L}_2(H)} < \infty$ or $\sum_j \lambda_j^{\kappa-1}\mu_j < \infty$
3. G is a linear operator on the spectrum of A:

$$(Gu)v = \sum_{j=1}^{\infty} \zeta_j \langle u, e_{j+m}\rangle \langle v, e_j\rangle e_j,$$

for a shift $m \in \mathbb{N}_0$ and a sequence $(\zeta_j)_{j=1}^\infty \subset \mathbb{R}$ fulfilling

$$|\zeta_j|\mu_j^{1/2} \leq C\lambda_j^{\frac{1-\kappa-\delta}{2}},$$

for some $1 \gg \delta > 0$, and ensuring that $G \in \mathcal{L}(H, \mathcal{L}_2(Q^{1/2}(H), \dot{H}^{\kappa+\delta-1}))$. When $m \neq 0$, an Euler-Maruyama scheme is *not* equivalent to a Milstein one (otherwise we would obtain improved convergence rate w.r.t. M).

Assumptions on F

Original assumptions[8]: $F\colon H \to H$ is twice Fréchet differentiable and

$$\begin{aligned} \|F'(u)v\|_{\dot H^{-r}} &\le C\|v\|_{\dot H^{-r}}, \quad \text{for } r \in \{0,1,2\} \text{ and all } u, v \in H \\ \|F''(u)(v,w)\|_{\dot H^{-2}} &\le C\|v\|_{\dot H^{-1}}\|w\|_{\dot H^{-1}}, \text{ for all } u, v \in H \end{aligned}$$

for C ind. of u. If F is a composition operator in $L^2(\mathcal{D})$, *this implies linearity*.

Instead: Let $F \in \mathcal{G}^1(H,H) \cap \mathcal{G}^2(H, \dot H^{-\eta})$ for some $\eta \in [0,2)$.

- (i) $\|F'(u)v\| \le C\|v\|$ for all $u, v \in H$,
- (ii) $\|F(u) - F(v)\|_{\dot H^{\kappa}} \le C(1 + \|u\|^2_{\dot H^{\kappa}} + \|v\|^2_{\dot H^{\kappa}})$ for all $u, v \in \dot H^{\kappa}$,
- (iii) $\|F'(u)v\|_{\dot H^{-\eta}} \le C(1 + \|u\|^{\lceil \kappa \rceil}_{\dot H^{\kappa}})\|v\|_{\dot H^{-\kappa}}$ for all $u \in \dot H^{\kappa}, v \in H$ and
- (iv) $\|F''(u)(v_1, v_2)\|_{\dot H^{-\eta}} \le C\|v_1\|\|v_2\|$ for all $u, v_1, v_2 \in H$.

[8] Jentzen, A. & Kloeden, P. E. Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise. English. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* **465,** 649–667 (2009).

Computation

The exponential integrator approximation X_N^M of the SPDE is now given by $X_N^M(0) := P_N X_0$ and for $j \in \{0, \ldots, M-1\}, \Delta t = T/M, t_j = j\Delta t$, by

$$X_N^M(t_{j+1}) := e^{-A\Delta t} X_N^M(t_j) + \int_{t_j}^{t_{j+1}} e^{-A(t_{j+1}-s)} P_N F(X_N^M(t_j)) \, \mathrm{d}s + \int_{t_j}^{t_{j+1}} e^{-A(t_{j+1}-s)} \big(P_N + G X_N^M(t_j)\big) \, \mathrm{d}W(s).$$

Note that

$$\Big\langle \int_{t_j}^{t_{j+1}} e^{-A(t_{j+1}-s)} \big(P_N + G X_N^M(t_j)\big) \, \mathrm{d}W(s), e_k \Big\rangle = \mu_k^{\frac{1}{2}} (1 + \alpha_k \langle X_N^M(t_j), e_{k+m} \rangle) \int_{t_j}^{t_{j+1}} e^{-\lambda_k (t_{j+1}-s)} \, \mathrm{d}B_k(s)$$

if $k \leq N$ and 0 otherwise, meaning the stochastic term can be sampled exactly.

Regularity and error estimates

Recall that $\lambda_k \eqsim k^\nu$ for some $\nu > 0$. Given the assumptions above:

Theorem

For $\bar{N} \geq N, \bar{M} \geq M$ and a known $C_N, C > 0$

- (i) $\sup_t \|X_N^M(t)\|^2_{L^p(\Omega,\dot{H}^\kappa)} < \infty$
- (ii) $\sup_t \|X_{\bar{N}}^M(t) - X_N^M(t))\|^2_{L^p(\Omega,H)} \leq C\lambda_{N+1}^{-\kappa} \eqsim N^{-\nu\kappa}$,
- (iii) $\sup_t \|X_N^{\bar{M}}(t) - X_N^M(t)\|^2_{L^p(\Omega,H)} \leq \min(CM^{-\min(\kappa,1)}, C_N M^{-1})$ *and*
- (iv) $$\sup_t \|X_{\bar{N}}^{\bar{M}}(t) - X_N^{\bar{M}}(t) - X_{\bar{N}}^M(t) + X_N^M(t)\|^2_{L^2(\Omega,H)}$$
$$\leq C\lambda_N^{-\kappa} \begin{cases} M^{-\kappa}\min(M^{-\kappa},1) & \kappa \in (0,1/2) \\ M^{-\kappa}\min(M^{\kappa-1}\lambda_N^{1-\kappa},1) & \kappa \in [1/2,1) \\ M^{-1} & \kappa \in [1,2) \end{cases}$$

yields $$\mathbb{E}[\|\Delta_\ell \Psi(X)\|^2_U] \lesssim \begin{cases} 2^{-\kappa\,\ell_1 - \kappa\nu\,\ell_2 - \kappa\max(\ell_1 - \nu\ell_2)} & \kappa \in (0,1/2) \\ 2^{-\kappa\,\ell_1 - \kappa\nu\,\ell_2 - (1-\kappa)\max(\ell_1-\nu\ell_2,0)} & \kappa \in [1/2,1) \\ 2^{-\ell_1 - \kappa\nu\,\ell_2} & \kappa \in [1,2) \end{cases}$$

Proof ideas

$$
\begin{aligned}
X_{\bar N}^{\bar M}(t) - X_N^{\bar M}(t) - X_{\bar N}^{M}(t) + X_N^M(t) \\
&= e^{-At}(P_{\bar N} - P_N - P_{\bar N} + P_N)X_0 \\
&\quad + \int_0^t e^{-A(t-s)}[\text{second order difference for } PF](s)\,\mathrm{d}s \\
&\quad + \int_0^t e^{-A(t-s)}[\text{second order difference for } PG](s)\,\mathrm{d}W(s) \\
&\quad + \int_0^t e^{-A(t-s)}(P_{\bar N} - P_N - P_{\bar N} + P_N)\,\mathrm{d}W(s).
\end{aligned}
$$

Neither the initial term nor the additive stochastic term directly contribute to either the spatial or the temporal part of the error!

Proof ideas

Focus on PF:

$$
\begin{aligned}
&P_{\bar N}F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}})) - P_N F(X_N^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}})) - P_{\bar N}F(X_{\bar N}^{M}(\lfloor s\rfloor_{M^{-1}})) + P_N F(X_N^{M}(\lfloor s\rfloor_{M^{-1}}))\\
&= P_{\bar N}(I-P_N)(F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}})) - F(X_{\bar N}^{M}(\lfloor s\rfloor_{M^{-1}})))\\
&+ P_N\Big(F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{M^{-1}})) - F(X_N^{\bar M}(\lfloor s\rfloor_{M^{-1}})) - F(X_{\bar N}^{M}(\lfloor s\rfloor_{M^{-1}})) + F(X_N^{M}(\lfloor s\rfloor_{M^{-1}}))\Big)\\
&+ P_N\Big(F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}})) - F(X_N^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}})) - F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{M^{-1}})) + F(X_N^{\bar M}(\lfloor s\rfloor_{M^{-1}}))\Big)
\end{aligned}
$$

Then

$$
\left\|\int_0^t e^{-A(t-s)}P_{\bar N}(I-P_N)(F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}})) - F(X_{\bar N}^{M}(\lfloor s\rfloor_{M^{-1}})))ds\right\|_{L^2(\Omega,H)}
$$

$$
\lesssim \int_0^t \|(I-P_N)A^{-\frac{\kappa}{2}}\|_{\mathcal{L}(H)}\|A^{\frac{\kappa}{2}}e^{-A(t-s)}\|_{\mathcal{L}(H)}\,\|(F(X_{\bar N}^{\bar M}(\lfloor s\rfloor_{\bar M^{-1}}))-F(X_{\bar N}^{M}(\lfloor s\rfloor_{M^{-1}})))\|_{L^2(\Omega,H)}ds
$$

Here

$$
\|(I-P_N)A^{-\frac{\kappa}{2}}\|_{\mathcal{L}(H)} = \|(I-P_N)\|_{\mathcal{L}(\dot H^{\kappa},H)} \le \lambda_{N+1}^{-\kappa/2}
$$

and critically

$$
\|A^{\frac{\kappa}{2}}e^{-A(t-s)}\|_{\mathcal{L}(H)} = \|e^{-A(t-s)}\|_{\mathcal{L}(H,\dot H^{\kappa})} \lesssim (t-s)^{-\kappa/2}
$$

Proof ideas

Repeated use of the mean value theorem, e.g.,

$$
\begin{aligned}
&F(X_{\bar{N}}^{\bar{M}}(\lfloor s \rfloor_{M^{-1}})) - F(X_{N}^{\bar{M}}(\lfloor s \rfloor_{M^{-1}})) - F(X_{\bar{N}}^{M}(\lfloor s \rfloor_{M^{-1}})) + F(X_{N}^{M}(\lfloor s \rfloor_{M^{-1}})) \\
&\quad = \int_0^1 F'(\dots) \left(X_{\bar{N}}^{\bar{M}}(\lfloor s \rfloor_{M^{-1}}) - X_{N}^{\bar{M}}(\lfloor s \rfloor_{M^{-1}}) - X_{\bar{N}}^{M}(\lfloor s \rfloor_{M^{-1}}) + X_{N}^{M}(\lfloor s \rfloor_{M^{-1}}) \right) \mathrm{d}\lambda \\
&\qquad + \int_0^1 \int_0^1 F''(\dots) \big(X_{N}^{\bar{M}}(\lfloor s \rfloor_{M^{-1}}) - X_{N}^{M}(\lfloor s \rfloor_{M^{-1}}) \big) \big(X_{\bar{N}}^{M}(\lfloor s \rfloor_{M^{-1}}) - X_{N}^{M}(\lfloor s \rfloor_{M^{-1}}) \big) \, \mathrm{d}\lambda \, \mathrm{d}\tilde{\lambda},
\end{aligned}
$$

Using single-difference bounds, and use BDG inequality to deal with G, conclude with Grönwall inequality.

We derive sharper rates for $\mathbb{E}[\|\Delta_\ell \Psi(X)\|_U^2]$ when Ψ is linear.

Summary of Computational Complexities, $\kappa \in [1, 2]$

For different Monte Carlo based methods, the cost is $\mathcal{O}(\varepsilon^{-2-2u} |\log(\varepsilon^{-1})|^r)$

- For Monte Carlo, $r = 0$ and

$$2u = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \leq 2\left(1 + \frac{1}{\kappa\nu}\right)$$

- For MLMC, $r \leq 2 + 2u$ and

$$2u = \max\left(0, \frac{1/(\kappa\nu)}{\min(\alpha_1, \alpha_2/(\kappa\nu))}\right) \leq \frac{2}{\kappa\nu}$$

- For MIMC

$$2u = \begin{cases} 0 & \kappa\nu \geq 1, \\ \frac{1-\kappa\nu}{\alpha_2} & \kappa\nu < 1. \end{cases} \leq \begin{cases} 0 & \kappa\nu \geq 1, \\ 2\left(\frac{1}{\kappa\nu} - 1\right) & \kappa\nu < 1. \end{cases}$$

and

$$r = \begin{cases} 2 & \kappa\nu > 1, \\ 4 & \kappa\nu = 1, \\ 0 & \kappa\nu < 1. \end{cases}$$

Summary of Computational Complexities, $\kappa \in [0, 1]$

For different Monte Carlo based methods, the cost is $\mathcal{O}(\varepsilon^{-2-2u} |\log(\varepsilon^{-1})|^r)$

- For Monte Carlo, $r = 0$ and

$$2u = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} \leq 2\left(\frac{1}{\kappa} + \frac{1}{\kappa\nu}\right)$$

- For MLMC, $r \leq 2 + 2u$ and

$$\begin{aligned} 2u &= \max\left(0, \frac{1 + \min(1, 2\kappa)/(\kappa\nu) - \min(1, 2\kappa)}{\min(\alpha_1, \alpha_2 \min(1, 2\kappa)/(\kappa\nu))}\right) \\ &\leq 2 \max\left(0, \frac{1}{\kappa} + \frac{1}{\kappa\nu} - 1\right) \end{aligned}$$

- For MIMC, $r \leq 4 + 2u$ and

$$\begin{aligned} 2u &= \max\left(0, \frac{1 - \min(1, 2\kappa)}{\alpha_1 + \kappa/2}, \frac{1 - \kappa\nu}{\alpha_2}, \frac{1 + \nu(1 - 2\kappa)}{\alpha_2 + \nu\alpha_1}\right) \\ &\leq 2 \max\left(0, \frac{1}{2\kappa} - 1, \frac{1}{\kappa\nu} - 1, \frac{1 + \nu(1 - 2\kappa)}{\kappa\nu}\right) \end{aligned}$$

Overview

Verification of multiplicative convergence rate

Test for QoI $\Psi(x) = x$ and seek to verify multiplicative convergence property

$$\|\Delta_{\boldsymbol{\ell}} X\|_{L^2(\Omega,H)} \approx \sqrt{E_{M=10^4}[\|\Delta_{\boldsymbol{\ell}} X\|^2]} =: e(\ell_1, \ell_2)$$

with $M_\ell \eqsim N_\ell \eqsim 2^\ell$.
When $\kappa \geq 1$, our sharp theoretical rates (for linear Ψ) are:

$$\|\Delta_{\boldsymbol{\ell}} X\|^2_{L^2(\Omega,H)} \approx C \min(M_{\ell_1}^{-1} N_{\ell_2}^{-\nu\kappa}, N_{\ell_2}^{-2\nu\kappa})$$

Numerical verification: Find $\beta_1, \beta_2 > 0$ by a least square fit, such that

$$p^2(\ell_1, \ell_2) := C \min(2^{-\beta_1 \ell_1 - \beta_2 \ell_2}, 2^{-2\beta_2 \ell_2})$$

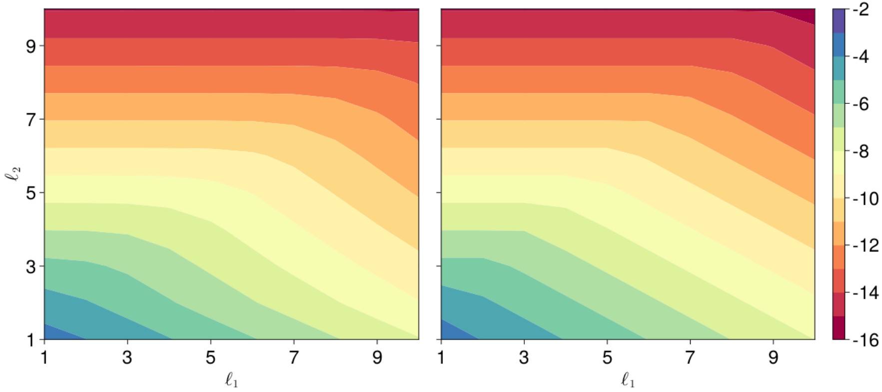
dominates $e(\ell_1, \ell_2)$ and verify that $\beta_1 \approx 1, \beta_2 \approx \nu\kappa$, when $\kappa \approx 1$ and $\nu = 4/3$.

For plotting, $\log_2(p(\ell_1, \ell_2))$ for a product $p(\ell_1, \ell_2)$ would be a plane over (ℓ_1, ℓ_2).

Note: Monte Carlo cost is $\mathcal{O}(\varepsilon^{-5.5})$, MLMC cost is $\mathcal{O}(\varepsilon^{-3.5})$.

Numerical test I

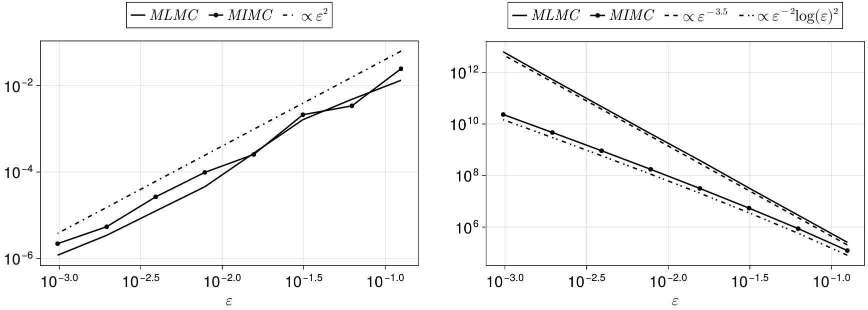
SPDE with $A = 0.2(-\Delta)^{2/3}$ on $\mathcal{D} = (0,1)$. Choose Q such that $\kappa < 1.01$ and let $f(x) = x$. Plot e and p in loglog scale.



Left: $e(\ell_1, \ell_2)$. **Right:** $p(\ell_1, \ell_2)$ with $\beta_1 = 0.98$, $\beta_2 = 1.62$.
Expected rates: $\beta_1 = 1$ and $\beta_2 = 4/3$.

Numerical test II

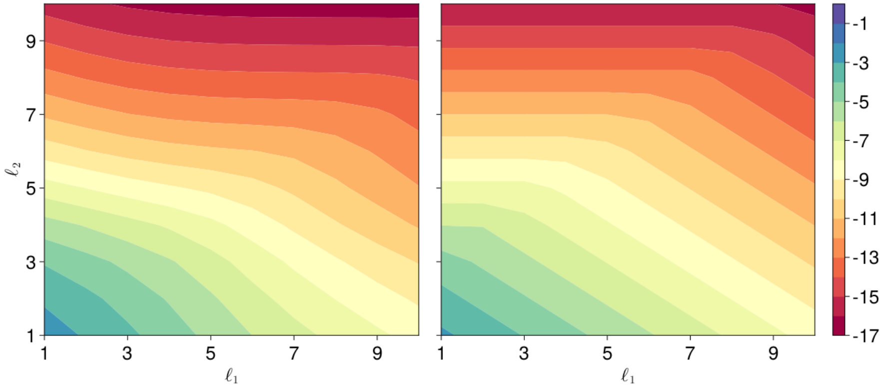
Performance comparison MLMC vs MIMC, linear case.



Left: Error $\|\mu_{\mathsf{MI}}(\varepsilon) - \mathbb{E}[X(1)]\|^2$. **Right:** $\mathrm{Cost}(\mu_{\mathsf{MI}}(\varepsilon))$

Numerical test III

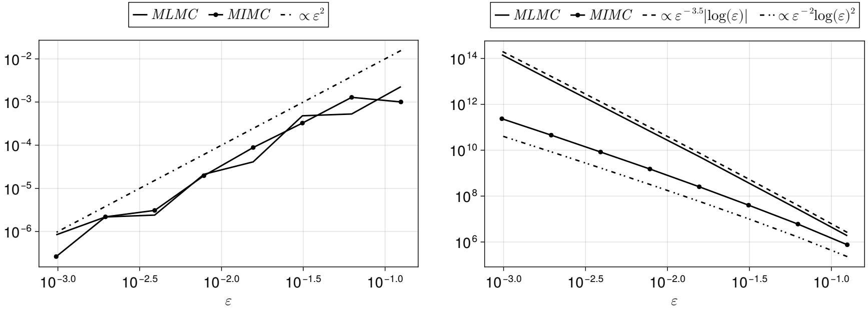
SPDE with $A = (-\Delta)^{2/3}$ on $\mathcal{D} = (0, 1)$. Choose Q such that $\kappa < 1.01$ and let $f(x) = \sin(\pi x)$.



Left: $e(\ell_1, \ell_2)$. **Right:** $p(\ell_1, \ell_2)$ with $\beta_1 = 1.24$, $\beta_2 = 1.74$.
Expected rates: $\beta_1 = 1$ and $\beta_2 = 4/3$.

Numerical test IV

Performance comparison MLMC vs MIMC, nonlinear case.



Left: Error $\|\mu_{\mathsf{MI}}(\varepsilon) - \mathbb{E}[X(1)]\|^2$. **Right:** $\mathrm{Cost}(\mu_{\mathsf{MI}}(\varepsilon))$

Summary and future work

- Developed efficent multi-index Monte Carlo method for approximations of semilinear SPDE
- We obtain high convergence in space and can handle sufficiently differentiable composition mappings F
- Restriction: Operators Q and A have to share eigenbasis (e_k) on which G acts
- Future work: Extension to finite element exponential integrators
- Future work: Nonlinear G acting on the eigenbasis
- Future work: Sharp rates in time for $G = 0$ using stochastic sewing[9]

[9] Djurdjevac, A., Gerencsér, M. & Kremp, H. Higher order approximation of nonlinear SPDEs with additive space-time white noise. *arXiv preprint arXiv:2406.03058.* arXiv: 2406.03058 [math.PR]. https://doi.org/10.48550/arXiv.2406.03058 (June 2024).

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- Chada, N. K., Hoel, H., Jasra, A. & Zouraris, G. E. Improved efficiency of multilevel Monte Carlo for stochastic PDE through strong pairwise coupling. *J. Sci. Comput.* **93,** Paper No. 62, 29 (2022)

Other notions of solutions

An H-valued predictable process $\{X(t)\}_{t\in[0,T]}$ is called:

- a **strong solution** of the SPDE if for all $t \in (0, T]$,

$$X(t) = X_0 + \int_0^t F(X(s)) - AX(s)\,\mathrm{d}s + \int_0^t (I + GX(s))\,\mathrm{d}W(s).$$

  Problem: Need that $X \in \mathrm{Dom}(A)$, and often it is not that smooth.

- a **weak solution** of the SPDE if for all $t \in (0, T]$ and $v \in \mathrm{Dom}(A)$

$$\begin{aligned}\langle X(t), v\rangle =&\langle X_0, v\rangle + \int_0^t \langle F(X(s)), v\rangle - \langle X(s), Av\rangle\,\mathrm{d}s \\ &+ \int_0^t \langle (I + GX(s))\,\mathrm{d}W(s), v\rangle.\end{aligned}$$

Relationship[10]**:** Strong solutions are weak solutions and weak solutions are typically mild solutions.

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[10] Liu, W. & Röckner, M. *Stochastic partial differential equations: an introduction.* (Springer, 2015).

Motivation

Regularity: The semigroup e^{-At} is smoothing:

$$\|e^{-At}\|_{\mathcal{L}(H)} = \|e^{-At}e_1\| = e^{-\lambda_1 t} < 1$$

So that

$$\|e^{-At}X_0\|_{\dot{H}^1} = \|A^{1/2}e^{-At}X_0\| \leq \|e^{-At}A^{1/2}X_0\| \leq \|A^{1/2}X_0\|.$$

And $\|A^{1/2}e^{-At}\|_{\mathcal{L}(H)} \leq Ct^{-1/2}$ used to bound $\dot{H}^1$-norm of other terms, e.g.,

$$\begin{aligned}\int_0^t \|e^{-A(t-s)}P_N F(X_N^M(s))\|_{L^p(\Omega,\dot{H}^1)}\,\mathrm{d}s \\ &= \int_0^t \|A^{1/2}e^{-A(t-s)}P_N F(X_N^M(s))\|_{L^p(\Omega,H)}\,\mathrm{d}s \\ &\leq c\int_0^t (t-s)^{-1/2}\|P_N F(X_N^M(s))\|_{L^p(\Omega,H)}\,\mathrm{d}s\end{aligned}$$

Motivation

Numerical error: bound by

$$\|X(T) - X_N^M(T)\| \leq \underbrace{\|X(T) - P_N X(T)\|}_{\text{spatial error}} + \underbrace{\|P_N X(T) - X_N^M(T)\|}_{\text{time error}}.$$

Spatial error:

$$\|(I - P_N)X(T)\| = \|A^{-1/2}(I - P_N)A^{1/2}X(T)\| \leq \|A^{-1/2}(I - P_N)\|_{\mathcal{L}(H)}$$

and

$$\|A^{-1/2}(I - P_N)\|_{\mathcal{L}(H)} = \|A^{-1/2}(I - P_N)e_{N+1}\| = \lambda_{N+1}^{-1/2}$$

Time error:

$$\|P_N X(T) - X_N^M(T)\|_{L^p(\Omega,H)} = \mathcal{O}(\sqrt{\Delta t})$$

is same rate as Euler–Maruyama has for N-dimensional SDE.

Negative norm bounds

Negative norm bounds on F,

$$\|F'(u)v\|_{\dot{H}^{-\eta}} \leq C(1+\|u\|_{\dot{H}^{\kappa}}^{\lceil r \rceil})\|v\|_{\dot{H}^{-r}}, \quad r \in \{1, \kappa-\delta\}, \quad u \in \dot{H}^{\kappa}, v \in H,$$

follow from a duality argument applied to

$$\|F'(u)v\|_{\dot{H}^{r}} \leq C(1+\|u\|_{\dot{H}^{\kappa}}^{\lceil r \rceil})\|v\|_{\dot{H}^{\eta}}, \quad r \in \{1, \kappa-\delta\}, \quad u \in \dot{H}^{\kappa}, v \in \dot{H}^{\eta}$$

in the case that $F'(u)$ is symmetric on H.
Case that $A = -\Delta$ on $H = L^2(\mathcal{D})$ with zero Dirichlet b.c. and F is a composition mapping: Identify $\dot{H}^r$ with $W^{r,2}$ or $W_0^{r,2}$ and note $(F'(u)v)(\cdot) = f'(u(\cdot))v(\cdot)$.
Use Sobolev embedding and multiplication theorems with $\eta \in (d/2, 2)$ to deduce

Lemma

For $\kappa \leq \min(\eta, 1)$ *and* f *twice differentiable with bounded derivatives:*

$$\|F'(u)v\|_{W^{\kappa,2}} \lesssim \|u\|_{W^{\kappa,2}}\|v\|_{W^{\eta,2}} \quad u \in W^{\kappa,2}, v \in W^{\eta,2}.$$

For $\kappa \in (1, \eta), d \leq 2$ *and* f *thrice differentiable with bounded derivatives:*

$$\|F'(u)v\|_{W^{\kappa-\delta,2}} \lesssim (1+\|u\|_{W^{\kappa,2}}^2)\|v\|_{W^{\eta,2}} \quad u \in W^{\kappa,2}, v \in W^{\eta,2}, \delta > 0.$$