Multilevel Monte Carlo and Path Branching for Digital Options

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The problem: Pricing a Digital option

Let X_t be a d-dimensional stochastic process satisfying the SDE for $0 < t \leq 1$

$$\mathrm{d}X_t = a(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}W_t.$$

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We want to price a digital option of the form (dropping discounting)

$$\mathbb{P}[\,X_1 \in K\,] = \mathbb{E}[\,\mathbb{I}_{X_1\in K}\,]$$

for some $K \subset \mathbb{R}^d$. Let $\{\overline{X}_t^\ell\}_{t=0}^1$ be an approximation of the path $\{X_t\}_{t=0}^1$ at level ℓ using $h_\ell^{-1} \equiv 2^\ell$ timesteps.

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For $|\mathbb{E}[\, \mathbb{I}_{X_1 \in K} - \mathbb{I}_{\overline{X}^\ell_1 \in K} \,]| \lesssim h_\ell^\alpha$, a Monte Carlo estimator of $\mathbb{E}[\, \mathbb{I}_{X_1 \in K} \,]$ has computational complexity $\varepsilon^{-2-\alpha}$ to achieve MSE ε.

Multilevel Monte Carlo

Consider a hierarchy of corrections $\{\Delta P_\ell\}_{\ell=0}^{L}$ such that

$$\mathbb{E}[\,\Delta P_\ell\,] = \begin{cases} \mathbb{E}\left[\,\mathbb{I}_{\overline{X}_1^0 \in K}\,\right] & \ell = 0 \\ \mathbb{E}\left[\,\mathbb{I}_{\overline{X}_1^\ell \in K} - \mathbb{I}_{\overline{X}_1^{\ell-1} \in K}\,\right] & \text{otherwise.} \end{cases}$$

MLMC can be formulated as

$$\mathbb{E}\left[\,\mathbb{I}_{X_1 \in K}\,\right] = \sum_{\ell=0}^{\infty} \mathbb{E}[\,\Delta P_\ell\,] \approx \sum_{\ell=0}^{L} \frac{1}{M_\ell} \sum_{m=1}^{M} \Delta P_\ell^{(m)}$$

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Assuming

$$\mathrm{Var}[\,\Delta P_\ell\,] \lesssim h_\ell^{\beta_{\mathsf{d}}}, \qquad |\mathbb{E}[\,\Delta P_\ell\,]| \lesssim h_\ell^{\alpha}, \qquad \mathrm{Work}(\Delta P_\ell) \lesssim h_\ell^{-1}$$

then to compute with MSE ε^2 the complexity of MLMC is $\mathcal{O}(\varepsilon^{-2+\max((\beta_{\mathsf{d}}-1),0)/\alpha})$ when $\beta_{\mathsf{d}} \neq 1$ and $\mathcal{O}(\varepsilon^{-2}|\log \varepsilon|^2)$ otherwise.
E.g. Euler-Maruyama has $\alpha = 1$ and $\beta_{\mathsf{d}} \approx 1/2$ and complexity is $\mathcal{O}(\varepsilon^{-5/2})$.

Conditional Expectation and Path Splitting

For some $0 < \tau < 1$, let

$$\Delta Q_\ell := \mathbb{E}[\, \Delta P_\ell \,|\, \mathcal{F}_{1-\tau} \,].$$

$$\text{Note} \qquad \mathbb{E}[\, \Delta Q_\ell \,] = \mathbb{E}[\, \Delta P_\ell \,].$$

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Computing ΔQ_ℓ:

- In 1D, taking $\tau \equiv h_\ell$ and using Euler-Maruyama for the last step we know that the conditional distribution of ΔP_ℓ given $\mathcal{F}_{1-\tau}$ is Gaussian and we can compute ΔQ_ℓ exactly.
- More generally, for any method and any τ, we can use path splitting (Monte Carlo) with sufficient number of samples, leading to increased work.

See, e.g., Glasserman (2004) and Burgos & Giles (2012) for more information on this method (for computing options and sensitivities).

Path splitting to estimate ΔQ_ℓ

- When $\tau \to 0$, i.e., splitting late,

$$\mathrm{Var}[\,\Delta Q_\ell\,] \leq \mathbb{E}\Big[\,(\mathbb{E}[\,\Delta P_\ell \,|\, \mathcal{F}_{1-\tau}\,])^2\,\Big] = \mathbb{E}\Big[\,(\Delta P_\ell)^2\,\Big] = \mathcal{O}(h_\ell^{\beta_{\mathsf{d}}})$$

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Hence we want to take $\tau \to 1$, but the cost per inner sample increases; paths are approximated over $[1-\tau, 1]$ for every sample.

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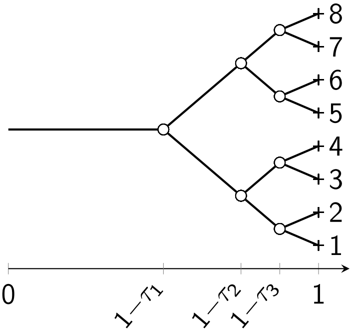
Solution: More splitting! For $\tau' > \tau$

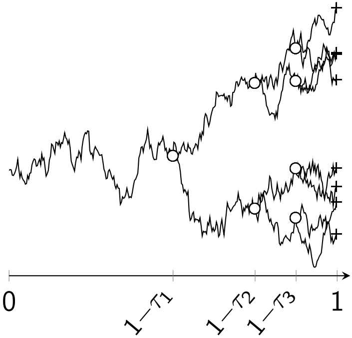
$$\begin{aligned}\Delta Q_\ell' &= \mathbb{E}[\,\Delta Q_\ell\,|\,\mathcal{F}_{1-\tau'}\,] \\ &= \mathbb{E}[\,\mathbb{E}[\,\Delta P_\ell\,|\,\mathcal{F}_{1-\tau}\,]\,|\,\mathcal{F}_{1-\tau'}\,]\end{aligned}$$

Again $\quad \mathbb{E}[\,\Delta Q_\ell'\,] = \mathbb{E}[\,\Delta P\,]$

Path Branching

- Let $1-\tau_{\ell'} = 1-2^{-\ell'}$ for $\ell' \in \{1,\ldots,\ell\}$.
- For every ℓ', starting from $X_{1-\tau_{\ell'}}$ at time $1-\tau_{\ell'}$, create two sample paths $\{X_t\}_{1-\tau_{\ell'} \le t \le 1-\tau_{\ell'+1}}$ which depend on two independent samples of the Brownian motion $\{W_t\}_{1-\tau_{\ell'} \le t \le 1-\tau_{\ell'+1}}$.
- Evaluate the payoff difference $\Delta P_\ell^{(i)}$ for every $X_1^{(i)}$ for $i \in \{1,\ldots,2^\ell\}$
- Define the Monte Carlo average as $\Delta \mathcal{P}_\ell := 2^{-\ell} \sum_{i=1}^{2^\ell} \Delta P_\ell^{(i)}$





Main Assumptions & Bounds

Assumption

Assume that there exists $\beta_{\mathsf{d}}, \beta_{\mathsf{c}}, p > 0$ such that for all $\tau > h_\ell$

$$\mathbb{E}[\,(\Delta P_\ell)^2\,] \lesssim h_\ell^{\beta_{\mathsf{d}}}$$

and $$\mathbb{E}\left[\,(\mathbb{E}[\,\Delta P_\ell \,|\, \mathcal{F}_{1-\tau}\,])^2\,\right] \lesssim h_\ell^{\beta_{\mathsf{c}}}/\tau^{1/2}$$

Theorem (Work/Variance bounds)

$$\mathbb{E}[\,\Delta \mathcal{P}_\ell\,] = \mathbb{E}[\,\Delta P_\ell\,]$$

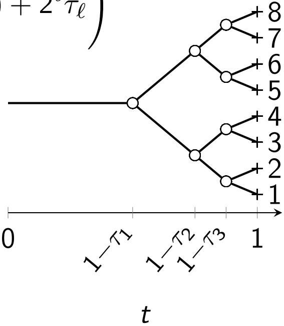
$$\mathrm{Work}(\Delta \mathcal{P}_\ell) \lesssim \ell\, h_\ell^{-1}$$

$$\mathrm{Var}[\,\Delta \mathcal{P}_\ell\,] \lesssim h_\ell^{\beta_{\mathsf{d}}+1} + h_\ell^{\beta_{\mathsf{c}}}$$

Proof

$$\text{Work}(\Delta\mathcal{P}_\ell) \le h_\ell^{-1}\left((1-\tau_1) + \sum_{\ell'=1}^{\ell-1} 2^{\ell'}(\tau_{\ell'} - \tau_{\ell'+1}) + 2^{\ell}\tau_\ell \right)$$

$$\lesssim \ell\, h_\ell^{-1}$$



$$\mathrm{Var}[\,\Delta\mathcal{P}_\ell\,] \le \mathbb{E}\left[\left(\frac{1}{2^\ell} \sum_{i=1}^{2^\ell} \Delta P_\ell^{(i)} \right)^2 \right]$$

$$\le \frac{1}{2^\ell}\mathbb{E}[\,\Delta P_\ell^2\,] + \frac{1}{2^{2\ell}} \sum_{i=1}^{\ell} \sum_{j=1, i\ne j}^{\ell} \mathbb{E}[\,\Delta P_\ell^{(i)} \Delta P_\ell^{(j)}\,]$$

$$\le \frac{1}{2^\ell}\mathbb{E}[\,\Delta P_\ell^2\,] + \frac{1}{2^{2\ell}} \sum_{i=1}^{\ell} \sum_{j=1, i\ne j}^{\ell} \mathbb{E}[\,(\mathbb{E}[\,\Delta P_\ell \,|\, \mathcal{F}_{1-\tau^{(i,j)}}\,])^2\,]$$

Simplified Assumptions on SDE solution/Approximation

Assumption (*On SDE solution*)

Assume that for some $\delta_0 > 0$ and all $0 < \delta \le \delta_0$ and $0 < \tau \le 1$, and let $d_{\partial K}(x) = \min_{y \in \partial K} \|x - y\|$, there is a constant C independent of δ, τ and $\mathcal{F}_{1-\tau}$ such that

$$\mathbb{E}\left[\left(\mathbb{P}[\, d_{\partial K}(X_1) \le \delta \,|\, \mathcal{F}_{1-\tau} \,]\right)^2 \right] \le C \, \frac{\delta^2}{\tau^{1/2}}.$$

Theorem

Assume additionally that there is $q > 2$ and $\beta > 0$ such that

$$\mathbb{E}\left[\left(X_1 - \overline{X}_1^\ell \right)^q \right]^{1/q} \lesssim h_\ell^{\beta/2}$$

Then $\quad \beta_{\mathrm{d}} = \dfrac{\beta}{2} \times \left(1 - \dfrac{1}{q+1}\right) \quad$ and $\quad \beta_{\mathrm{c}} = \beta \times \left(1 - \dfrac{2}{q+2}\right)$

MLMC Complexity

When q is arbitrary,

$$\beta_{\mathsf{d}} \approx \frac{\beta}{2} \qquad \text{and} \qquad \beta_{\mathsf{c}} \approx \beta$$

and for $\beta \leq 2$

$$\mathrm{Var}[\,\Delta\mathcal{P}_\ell\,] \approx \mathcal{O}(h_\ell^\beta)$$
$$\mathsf{Work}(\Delta\mathcal{P}_\ell) = \mathcal{O}(\ell h_\ell^{-1})$$

- Using Euler-Maryama: $\beta = 1$ and the MLMC computational complexity is approximately $o\left(\varepsilon^{-2+\nu}\right)$ for any $\nu > 0$ and for MSE ε.
- Using Milstein: $\beta = 2$ and the complexity is $\mathcal{O}(\varepsilon^{-2})$.

SDEs with Gaussian Transition Kernels

Lemma

Assume that a and σ are bounded and uniformly Hölder continuous and σ is uniformly elliptic then there is $C > 0$ such that

$$\mathbb{E}\Big[\big(\mathbb{P}[\, d_{\partial K}(X_1) \leq \delta \,|\, \mathcal{F}_{1-\tau} \,]\big)^2 \Big] \leq C \, \frac{\delta^2}{\tau^{1/2}}$$

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and

$$\mathbb{E}\Big[(\mathbb{P}[\, d_{\partial K}(\exp(X_1)) \leq \delta \,|\, \mathcal{F}_{1-\tau}\,])^2 \Big] \leq C\, \frac{\delta^2}{\tau^{1/2}}.$$

when ∂K are "nice".

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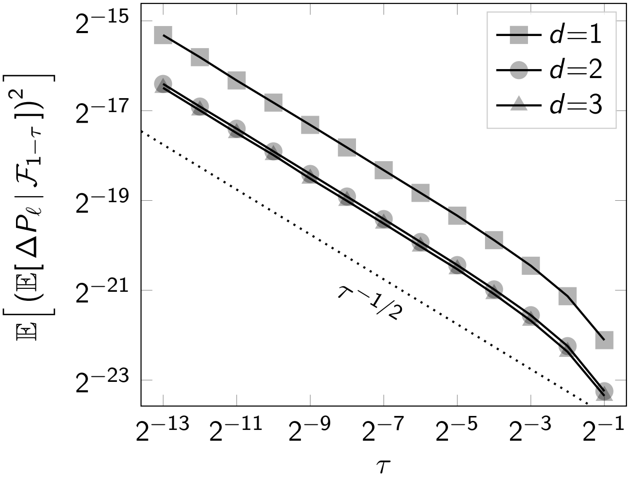
Proof. Based on bounding the conditional density of X_1 by a Gaussian density. E.g.

$$\mathbb{E}\left[\left(\mathbb{P}[\, d_{\partial K}(X_1) \le \delta \,|\, \mathcal{F}_{1-\tau} \,]\right)^2 \right]$$
$$\lesssim \frac{1}{\tau^{1/2}} \left(\int_{-\delta}^{\delta} \mathrm{d}x \right) \times \mathbb{E}[\, \mathbb{P}[\, d_{\partial K}(X_1) \le \delta \,|\, \mathcal{F}_{1-\tau} \,]\,] \lesssim \frac{\delta^2}{\tau^{1/2}}$$

Numerical Results on GBM



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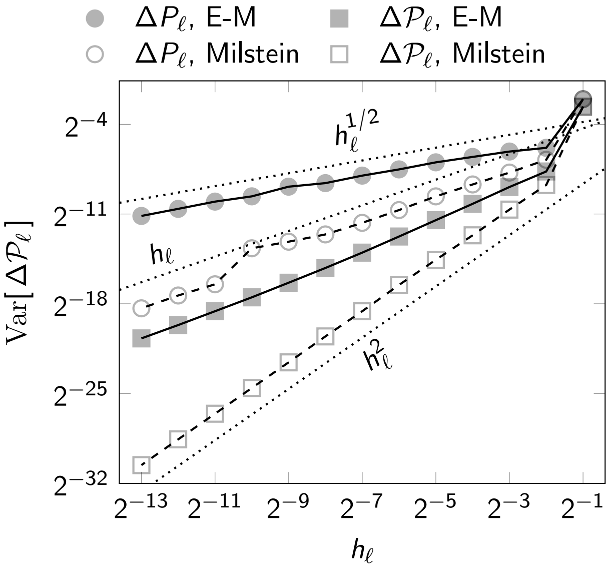

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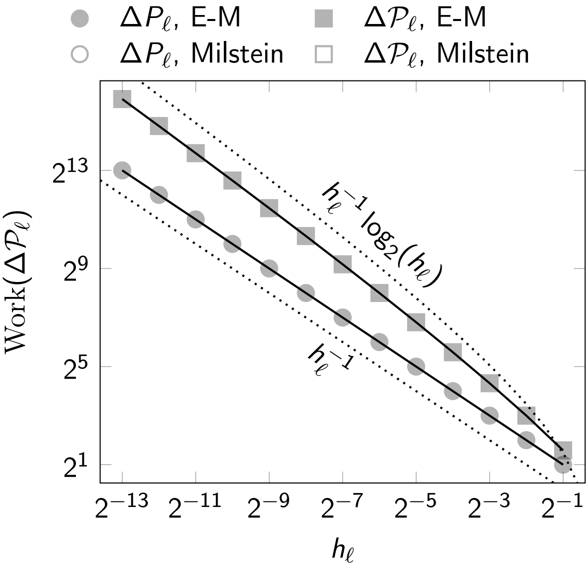
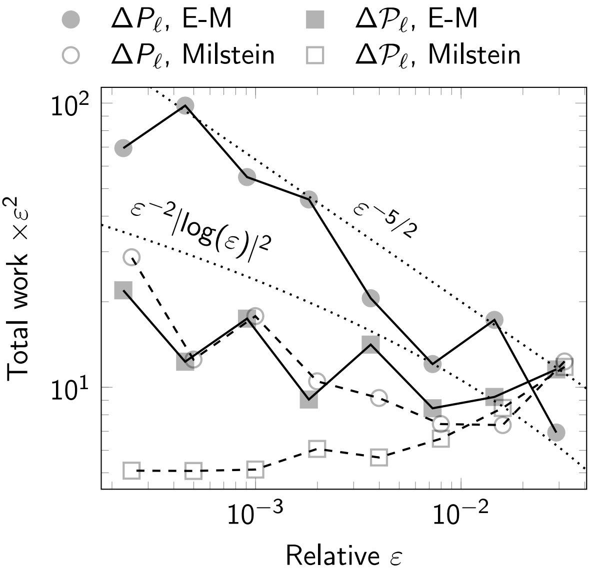
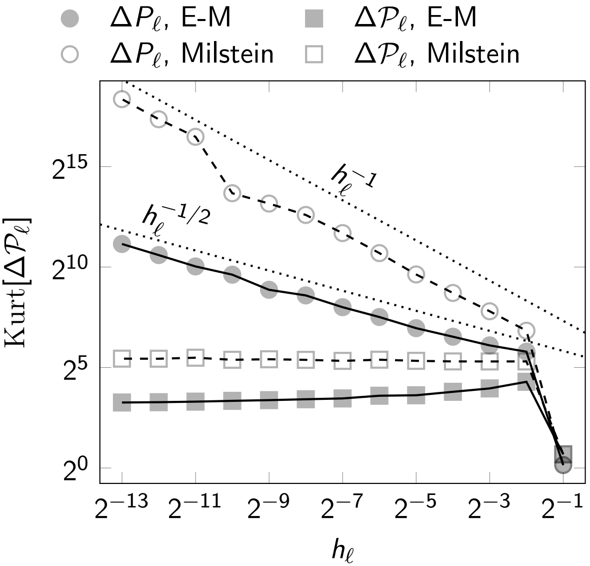



ΔP_ℓ, E-M
$\Delta \mathcal{P}_\ell$, E-M
ΔP_ℓ, Milstein
$\Delta \mathcal{P}_\ell$, Milstein
10^2
10^1
$\varepsilon^{-2}|\log(\varepsilon)|^2$
$\varepsilon^{-5/2}$
10^{-3}
10^{-2}
Total work $\times\varepsilon^2$
Relative ε

Numerical Results on GBM



Antithetic estimator

For the Clark-Cameron SDE ($\mathrm{d}X_t = W_{1,t}\,\mathrm{d}W_{2,t}$), using a Milstein scheme requires sampling Lévy areas.

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Giles & Szpruch (2014) proposed an antithetic Milstein scheme (with Lévy area set to zero). Applying to digital options we set

$$\Delta P_\ell = \begin{cases} \mathbb{I}_{\overline{X}_1^\ell \in K} & \ell = 0 \\ \frac{1}{2}\left(\mathbb{I}_{\overline{X}_1^\ell \in K} + \mathbb{I}_{\overline{X}_1^{\ell,(a)} \in K}\right) - \mathbb{I}_{\overline{X}_1^{\ell-1} \in K} & \ell > 0 \end{cases}$$

where $\overline{X}_1^\ell$ and $\overline{X}_1^{\ell,(a)}$ are an identically distributed antithetic pair.

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where $\overline{X}_1^\ell$ and $\overline{X}_1^{\ell,(a)}$ are an identically distributed antithetic pair.

We have for all $q > 2$

$$\mathbb{E}\left[\left\|X_1 - \overline{X}_1^\ell\right\|^q\right]^{1/q} \le C h_\ell^{1/2}$$

$$\text{and} \qquad \mathbb{E}\left[\left\|\frac{1}{2}(\overline{X}_1^\ell + \overline{X}_1^{\ell,(a)}) - \overline{X}_1^{\ell-1}\right\|^q\right]^{1/q} \le C h_\ell.$$

Antithetic estimator

Lemma (Antithetic rates)

Under certain assumptions on $g(x,t) = \mathbb{E}[\,\mathbb{I}_{X_1 \in K} \,|\, X_t = x\,]$ *and its derivatives, we have*

$$\mathbb{E}[\,(\Delta P_\ell)^2\,] \lesssim h_\ell^{1/2(1-1/(q+1))}$$

$$\textit{and} \qquad \mathbb{E}\Big[\,(\mathbb{E}[\,\Delta P_\ell \,|\, \mathcal{F}_{1-\tau}\,])^2\,\Big] \lesssim h_\ell^{2(1-5/(q+5))}/\tau^{3/2}.$$

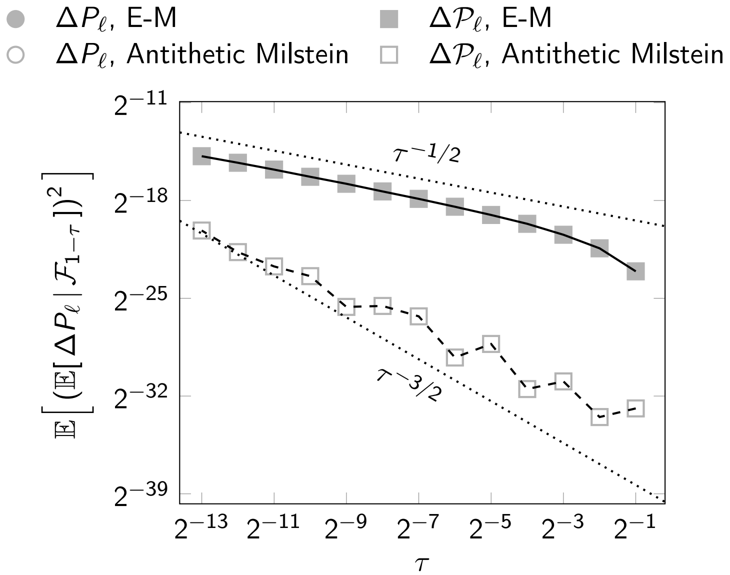
In other words

$$\beta_{\mathsf{d}} = \frac{1}{2} \times \left(1 - \frac{1}{q+1}\right) \qquad \text{and} \qquad \beta_{\mathsf{c}} = 2 \times \left(1 - \frac{5}{q+5}\right).$$

When q is arbitrary, we show that for any $\nu > 0$,

$$\mathrm{Var}[\,\Delta \mathcal{P}_\ell\,] \lesssim h_\ell^{3/2-\nu}$$

Numerical Results on Clark-Cameron



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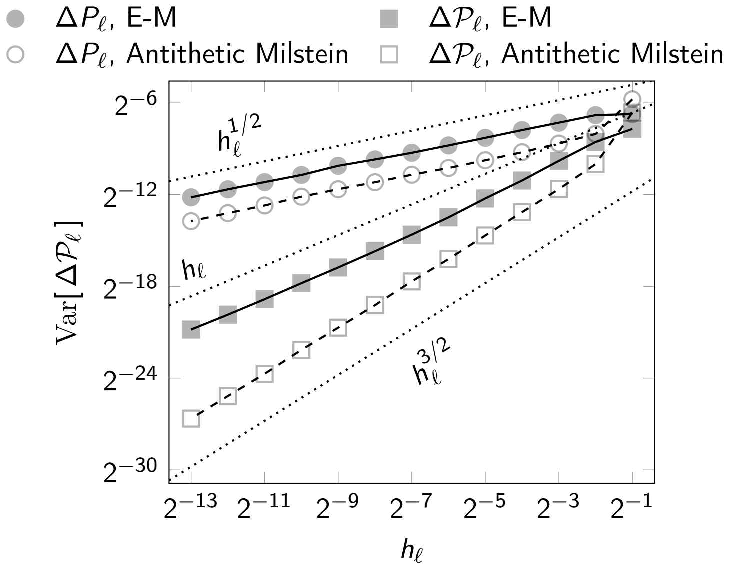

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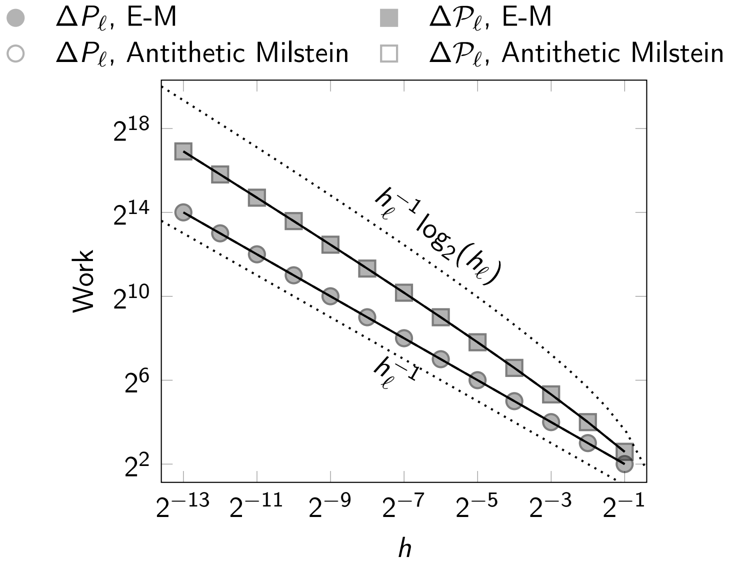

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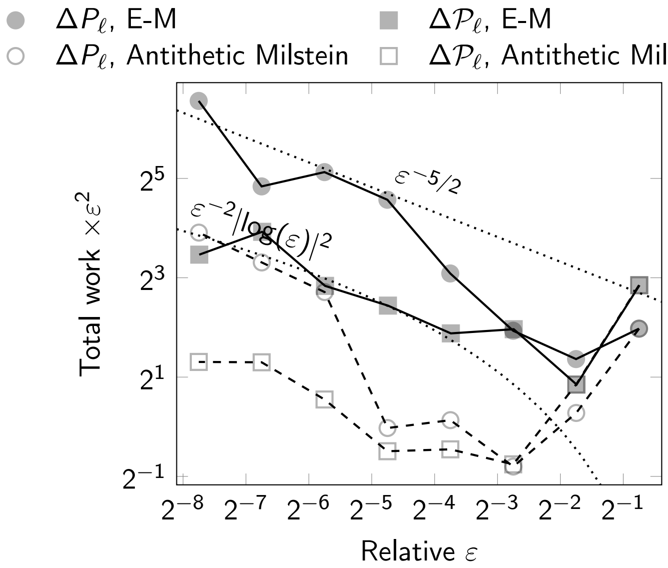

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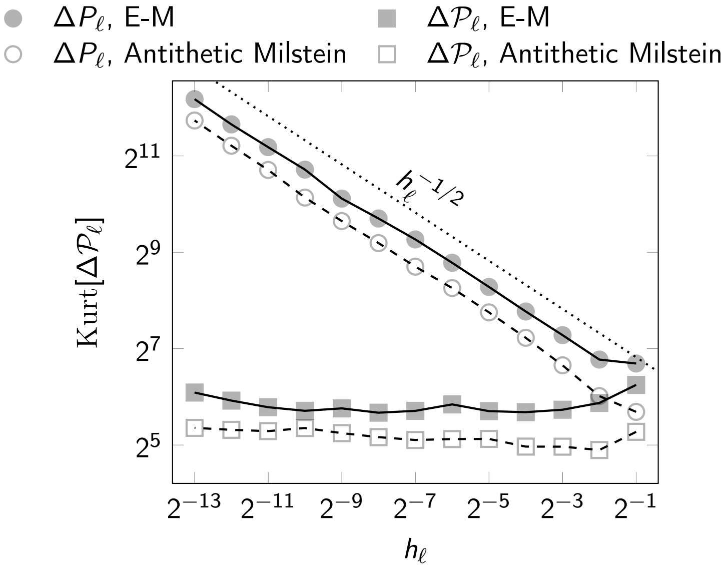


Final points

- We also consider a sequence $\tau_{\ell'} = 2^{-\eta \ell'}$ for some $\eta > 0$. For $\eta > 1$, this reduces the work of $\Delta \mathcal{P}_\ell$ to $\mathcal{O}(2^\ell)$.
- More theoretical and numerical analysis for antithetic estimators.
- A modular analysis: Application to other problems involving conditional expectations and filtrations can be done by proving assumptions.
- Other applications:
  - Pricing other options (Barrier).
  - Computing sensitivities: Using bumping, the variance increases as the bump distance decreases. Branching can help.
  - Particle systems.
  - Approximate CDFs. Need to tighten theory to deal with increasing number of discontinuities.
  - Parabolic SPDEs.

Elliptic SDEs

Definiton ((Ni) *sets*)

We say that a set $S \subset \mathbb{R}^d$ is an (Ni) set if there exists an orthonormal matrix A and a Lipschitz function f such that $S = A\widetilde{S}$ for the set

$$\widetilde{S} = \{x \in \mathbb{R}^d \ : \ f(x_{-1}) = x_1\},$$

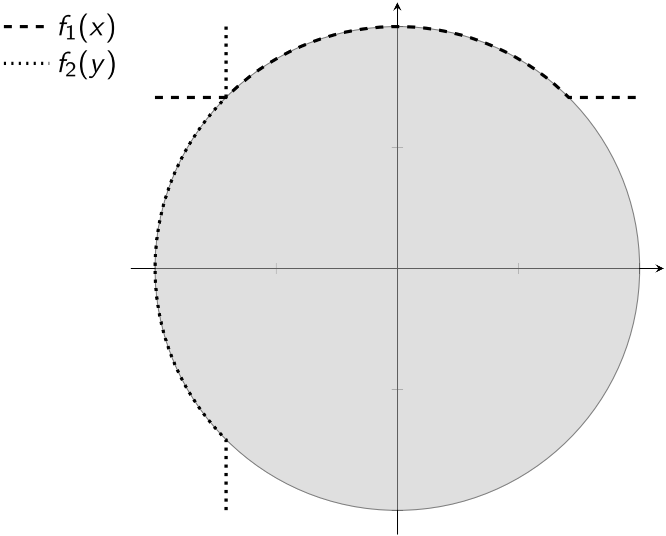
and $A\widetilde{S}$ denoting the image of $\widetilde{S}$ under the transformation $x \to Ax$.

Lemma

For $K \subset \mathbb{R}^d$ assume that $\partial K \subseteq \bigcup_{j=1}^n S_j$ for some finite n and (Ni) *sets $\{S_j\}_{j=1}^n$. Assume further that a and σ are bounded and uniformly Hölder continuous and σ is uniformly elliptic then*

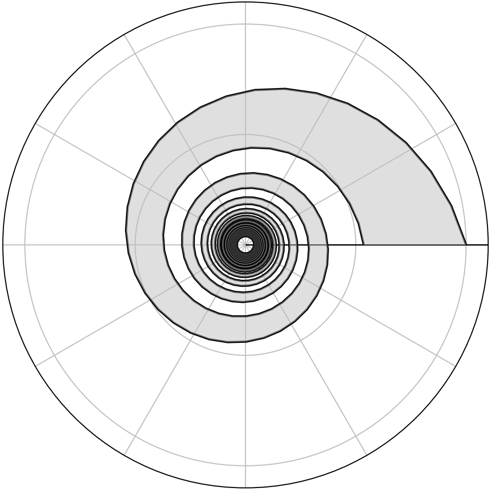
$$\mathbb{E}\Big[\left(\mathbb{P}[\, d_{\partial K}(X_1) \leq \delta \,|\, \mathcal{F}_{1-\tau}\,]\right)^2 \Big] \leq C\, \frac{\delta^2}{\tau^{1/2}}.$$

A nice set



$$\delta K = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$$

A not-so-nice set



$$\partial K = \{(r,\theta) \in \mathbb{R}_+ \times [0, 2\pi] \; : \; r = (n + \theta/\pi)^{-\frac{1}{2}}, n \in \mathbb{N}\}$$

Exponentials of Elliptic SDEs

What about a Geometric Brownian Motion $Y_t = \exp(X_t)$?

$$\mathrm{d}Y_t = aY_t\,\mathrm{d}t + \sigma Y_t\,\mathrm{d}W_t$$
$$\mathrm{d}X_t = a\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$$

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