Numerical Methods for Stochastic PDE
Lectures 1–2.
1. Stochastic evolution problems.
2. Strong convergence analysis.

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Outline

Stochastic heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f(u) + g(u)\dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0. \end{cases}$$

Stochastic wave equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f(u) + g(u)\dot{W}, & x \in \mathcal{D},\ t > 0 \\ u = 0, & x \in \partial\mathcal{D},\ t > 0 \\ u(0) = u_0,\ u_t(0) = u_1. \end{cases}$$

$\dot{W}$ is spatial and temporal noise

Outline

Stochastic Cahn–Hilliard equation (Cahn–Hilliard–Cook):

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta v = \dot{W} & \text{in } \mathcal{D} \times [0, T] \\ v = -\Delta u + f(u) & \text{in } \mathcal{D} \times [0, T] \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D} \times [0, T] \\ u(0) = u_0 & \text{in } \mathcal{D} \end{cases}$$

$f(u) = u^3 - u$

Outline

I will emphasize:

- abstract framework (functional analysis)
- the semigroup approach (mild formulation)
- existence, uniqueness, and regularity of solutions
- proof techniques
- spatial finite element discretization
- strong convergence of numerical approximations
- weak convergence of numerical approximations

Lecture 1. Stochastic evolution problem

Outline

Formulate as an abstract evolution problem in Hilbert space $\mathcal{H}$:

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W, \quad t > 0 \\ X(0) = X_0 \end{cases}$$

What does this mean? Strong formulation / variational formulation (depending on how regular X is assumed to be):

$$X(t) = X_0 + \int_0^t (-AX + F(X))\,\mathrm{d}s + \int_0^t G(X)\,\mathrm{d}W$$

Weak formulation:

$$\begin{aligned} \langle X(t), \eta \rangle = \langle X_0, \eta \rangle + \int_0^t \langle X(s), -A^* \eta \rangle + \langle F(X(s)), \eta \rangle \,\mathrm{d}s \\ + \int_0^t \langle \eta, G(X(s))\,\mathrm{d}W(s) \rangle \quad \forall \eta \in D(A^*) \end{aligned}$$

Outline

We will use the semigroup approach of Da Prato and Zabczyk [1] based on the mild formulation:

$$X(t) = \mathrm{e}^{-tA}X_0 + \int_0^t \mathrm{e}^{-(t-s)A}F(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A}G(X(s))\,\mathrm{d}W(s)$$

Here $\{\mathrm{e}^{-tA}\}_{t\geq 0}$ is the semigroup of bounded linear operators generated by $-A$.
$\{W(t)\}_{t\geq 0}$ is a Q-Wiener process in another Hilbert space $\mathcal{U}$ and $\int_0^t \cdots \mathrm{d}W$ is a stochastic integral.
We often study the linear case, where $F(X) = f$, $G(X) = B$ are independent of X:

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t + B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

Here $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Often $f = 0$ for brevity.
Additive noise: $B\,\mathrm{d}W$. Multiplicative noise: $G(X)\,\mathrm{d}W$.

Notation

- $\mathcal{D} \subseteq \mathbf{R}^d$ spatial domain, bounded, convex, with polygonal boundary
- $H = L_2(\mathcal{D})$ Lebesgue space
- $H^s = H^s(\mathcal{D})$ Sobolev space, $H_0^1 = \{v \in H^1 : v = 0 \text{ on } \partial\mathcal{D}\}$
- $\mathcal{H}$, $\mathcal{U}$ real, separable Hilbert spaces
- $\mathcal{L}(\mathcal{U}, \mathcal{H})$ bounded linear operators, $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$

$$\|T\|_{\mathcal{L}(\mathcal{U},\mathcal{H})} = \sup_{u\in\mathcal{U}} \frac{\|Tu\|_{\mathcal{H}}}{\|u\|_{\mathcal{U}}}$$

- $\mathcal{L}_2(\mathcal{U}, \mathcal{H})$ Hilbert–Schmidt operators, $\mathrm{HS} = \mathcal{L}_2(\mathcal{H}) = \mathcal{L}_2(\mathcal{H}, \mathcal{H})$

$$\|T\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \|Te_j\|^2_{\mathcal{H}}, \quad \text{with } \{e_j\}_{j=1}^{\infty} \text{ an arbitrary ON-basis in } \mathcal{U}$$

$$\langle S, T\rangle_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \langle Se_j, Te_j\rangle_{\mathcal{H}}$$

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Note:

$$\|ST\|_{\mathcal{L}_2(\mathcal{H})} \leq \|S\|_{\mathcal{L}(\mathcal{H})} \|T\|_{\mathcal{L}_2(\mathcal{H})}$$

Semigroup

A family $\{E(t)\}_{t\geq 0} \subseteq \mathcal{L}(\mathcal{H})$ is a semigroup of bounded linear operators on $\mathcal{H}$, if

- $E(0) = I, \quad$ (identity operator)
- $E(t+s) = E(t)E(s),\ t, s \geq 0. \quad$ (semigroup property)

It is strongly continuous, or C_0, if

$$\lim_{t\to 0+} E(t)x = x \quad \forall x \in \mathcal{H}.$$

Then the generator of the semigroup is the linear operator G defined by

$$Gx = \lim_{t\to 0+} \frac{E(t)x - x}{t}, \quad D(G) = \{x \in \mathcal{H} : Gx \text{ exists}\}.$$

G is usually unbounded but densely defined and closed.

Semigroup

$u(t) = E(t)u_0$ solves the initial-value problem

$$u'(t) = Gu(t),\ t > 0; \quad u(0) = u_0,$$

if $u_0 \in D(G)$. Therefore, writing $E(t) = \mathrm{e}^{tG}$ is justified.

There are $M \geq 1$, $\omega \in \mathbf{R}$, such that

$$\|E(t)\|_{\mathcal{L}(\mathcal{H})} \leq M\mathrm{e}^{\omega t}, \quad t \geq 0.$$

Without loss of generality we assume $\omega = 0$ (a shift of the operator $G \mapsto G - \omega I$). Contraction semigroup if also $M = 1$.

If $E(t)$ is invertible, $E(t)^{-1} = E(-t)$, then $\{E(t)\}_{t\in\mathbf{R}}$ is a group.

The semigroup is analytic (holomorphic), if $E(t)$ extends to a complex analytic function $E(z)$ in a sector containing the positive real axis $\operatorname{Re} z > 0$. Then the derivative

$$E'(t)u_0 = \frac{\mathrm{d}}{\mathrm{d}t}E(t)u_0 = GE(t)u_0, \quad t > 0,$$

exists for all $u_0 \in \mathcal{H}$, not just for $u_0 \in D(G)$. Moreover,

$$\|E'(t)u_0\|_{\mathcal{H}} = \|GE(t)u_0\|_{\mathcal{H}} \leq Ct^{-1}\|u_0\|_{\mathcal{H}}, \quad t > 0. \tag{1}$$

The inequality (1) is characteristic for analytic semigroups.

Semigroup

On the other hand, we may start with a closed, densely defined, linear operator A and ask for conditions under which $G = -A$ generates a semigroup $E(t) = \mathrm{e}^{-tA}$, so that $u(t) = E(t)u_0$ solves

$$u'(t) + Au(t) = 0, \ t > 0; \quad u(0) = u_0.$$

Such theorems exist, which characterize the generators of strongly continuous (C_0) semigroups, analytic semigroups, and groups. For example, Hille-Yosida theorem, Lumer-Phillips theorem, Stone's theorem.

For analytic semigroups a characterization is given in terms of the resolvent bound

$$\|(z - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|z - \omega|}, \quad \text{for } \operatorname{Re} z < \omega, \tag{2}$$

with ω as above ($\omega = 0$ without loss of generality).

Semigroup

The non-homogeneous equation (strong solution)

$$u'(t) + Au(t) = f(t),\ t > 0; \quad u(0) = u_0.$$

is then solved by the variation of constants formula (Duhamel's principle) (mild solution):

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(s)\,\mathrm{d}s,$$

provided that f has some small amount of regularity. This is the basis for our semigroup approach to SPDE.

Proof.

Multiply $u'(s) + Au(s) = f(s)$ by the "integrating factor" $\Phi(s) = E(t-s) = \mathrm{e}^{-(t-s)A}$, $t > s$, and integrate. □

Dirichlet Laplacian

Let $\mathcal{D} \subseteq \mathbf{R}^d$ be a bounded, convex, polygonal domain. Then

- finite element meshes can be exactly fitted to $\partial\mathcal{D}$;
- we have elliptic regularity: $(-\Delta v = f$ in $\mathcal{D}$; $v = 0$ on $\partial\mathcal{D})$

$$\|v\|_{H^2(\mathcal{D})} \leq C\|\Delta v\|_{L_2(\mathcal{D})} \quad \forall v \in H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

Here $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian. In this way we avoid some technical difficulties associated with the finite element method in smooth domains.

Let $H = L_2(\mathcal{D})$ and $\Lambda = -\Delta$ with $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$. Then Λ is unbounded in H and self-adjoint with compact inverse Λ^{-1}. The spectral theorem gives eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \ \lambda_j \to \infty, \ \lambda_j \sim j^{2/d} \text{ as } j \to \infty$$

and a corresponding orthonormal (ON) basis of eigenvectors $\{\varphi_j\}_{j=1}^{\infty}$.

Laplacian

Parseval's identity:

$$v \in H, \quad v = \sum_{j=1}^{\infty} \hat{v}_j \varphi_j, \quad \hat{v}_j = \langle v, \varphi_j \rangle_H, \quad \|v\|_H^2 = \sum_{j=1}^{\infty} \hat{v}_j^2$$

Fractional powers:

$$\Lambda^\alpha v = \sum_{j=1}^{\infty} \lambda_j^\alpha \hat{v}_j \varphi_j, \quad \alpha \in \mathbf{R}$$

$$\|v\|_{\dot{H}^\alpha}^2 = \|\Lambda^{\alpha/2} v\|_H^2 = \sum_{j=1}^{\infty} \lambda_j^\alpha \hat{v}_j^2, \quad \alpha \in \mathbf{R}$$

$$\dot{H}^\alpha = \{v \in H : \|v\|_{\dot{H}^\alpha} < \infty\} = D(\Lambda^{\alpha/2}), \quad \alpha \geq 0$$

$$\dot{H}^{-\alpha} = \text{ closure of } H \text{ in the } \dot{H}^{-\alpha}\text{-norm}, \quad \alpha > 0$$

Then $\dot{H}^{-\alpha}$ can be identified with the dual space $(\dot{H}^\alpha)^*$.

Laplacian

The integer order spaces can be identified with standard Sobolev spaces.

Theorem

(i) $\dot{H}^1 = H_0^1(\mathcal{D})$ *with* $\|v\|_{\dot{H}^1} = \|\nabla v\|_{L_2(\mathcal{D})} \simeq \|v\|_{H^1(\mathcal{D})}\ \forall v \in \dot{H}^1$
(ii) $\dot{H}^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ *with* $\|v\|_{\dot{H}^2} = \|\Delta v\|_{L_2(\mathcal{D})} \simeq \|v\|_{H^2(\mathcal{D})}\ \forall v \in \dot{H}^2$

Proof.

A proof of this can be found in Thomée [4, Ch. 3]. The proof of (i) is based on the Poincaré inequality and the trace inequality. The proof of (ii) uses also the elliptic regularity. In general, we have only

$$H^2(\mathcal{D}) \cap H_0^1(\mathcal{D}) \subseteq \dot{H}^2$$

because, in a nonconvex polygonal domain for example, $\dot{H}^2 = D(\Lambda)$ may contain functions with corner singularities which are not in $H^2(\mathcal{D})$. □

Laplacian

We define the heat semigroup:

$$E(t)v = \mathrm{e}^{-t\Lambda} v = \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda_j t} \hat{v}_j \varphi_j.$$

It is analytic in the right half plane $\operatorname{Re} z > 0$. Important bounds:

$$\|E(t)v\|_H \le \|v\|_H, \quad t \ge 0, \tag{3}$$

$$\|D_t^k E(t)v\|_H \le C_k t^{-k} \|v\|_H, \quad t > 0,\ k \ge 0, \tag{4}$$

$$\|\Lambda^\alpha E(t)v\|_H \le C_\alpha t^{-\alpha} \|v\|_H, \quad t > 0,\ \alpha \ge 0, \tag{5}$$

$$\int_0^t \|\Lambda^{1/2} E(s)v\|_H^2 \,\mathrm{d}s \le \tfrac{1}{2} \|v\|_H^2, \quad t \ge 0. \tag{6}$$

Recall from (1) that (4) is characteristic for analytic semigroups; and so is (5). They mean that the operator $E(t)$ has a smoothing effect. The smoothing effect in (6) is true for the heat semigroup, but not for analytic semigroups in general.

Laplacian

Proof.

We use Parseval and $x^\alpha e^{-x} \le C_\alpha$ for $x \ge 0$:

$$\|\Lambda^\alpha E(t)v\|_H^2 = \sum_{j=1}^{\infty} \left(\lambda_j^\alpha e^{-\lambda_j t} \hat{v}_j\right)^2 = t^{-2\alpha} \sum_{j=1}^{\infty} (\lambda_j t)^{2\alpha} e^{-2\lambda_j t} \hat{v}_j^2$$

$$\le C_\alpha^2 t^{-2\alpha} \sum_{j=1}^{\infty} \hat{v}_j^2 = C_\alpha^2 t^{-2\alpha} \|v\|_H^2.$$

This proves (3) and (5). Similarly, for (6),

$$\int_0^t \|\Lambda^{1/2} E(s)v\|_H^2 \, ds = \int_0^t \sum_{j=1}^{\infty} \lambda_j e^{-2\lambda_j s} \hat{v}_j^2 \, ds$$

$$= \sum_{j=1}^{\infty} \int_0^t \lambda_j e^{-2\lambda_j s} \, ds \, \hat{v}_j^2 \le \tfrac{1}{2} \|v\|_H^2.$$

□

Laplacian

Remark. The above development based on the spectral representation of fractional powers and the heat semigroup carries over verbatim to more general self-adjoint elliptic operators:

$$\Lambda v = -\nabla \cdot (a(x)\nabla v) + c(x)v \quad \text{with } 0 < a_0 \leq a(x) \leq a_1,\ c(x) \geq 0,$$

for then we still have an ON basis of eigenvectors. For non-self-adjoint elliptic operators, the fractional powers and the semigroup may be constructed by means of an operator calculus based complex contour integration using the resolvent, see (2). The bounds (3) and (5) are part of the general theory and (6) can be proved by an energy argument if the operator satisfies the conditions of the Lax-Milgram lemma, for example,

$$\Lambda v = -\nabla \cdot (a(x)\nabla v) + b(x) \cdot \nabla v + c(x)v \quad \text{with } c(x) - \tfrac{1}{2}\nabla \cdot b(x) \geq 0,$$

so that

$$\langle \Lambda v, v\rangle_H \geq c\|v\|_{\dot{H}^1}^2.$$

See the following exercises.

Laplacian

Exercise 1. Prove (6) by the energy method: multiply

$$u'(t) + \Lambda u(t) = 0 \tag{7}$$

by $u(t)$ and integrate.

Exercise 2. Prove the special case $\alpha = \frac{1}{2}$ of (5) by the energy method: multiply (7) by $tu'(t)$ and integrate.

Random variable

Let $\mathcal{U}$ be a separable real Hilbert space and let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A random variable is a measurable mapping $f : \Omega \to \mathcal{U}$, i.e.,

$$f^{-1}(A) \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathcal{U}) \text{ (= the Borel sigma algebra in } \mathcal{U}\text{)}.$$

We define Lebesgue–Bochner spaces $L_p(\Omega, \mathcal{U})$:

$$\|f\|_{L_p(\Omega,\mathcal{U})} = \Big(\int_\Omega \|f(\omega)\|_{\mathcal{U}}^p \, \mathrm{d}\mathbf{P}(\omega) \Big)^{1/p} = (\mathbf{E}[\|f\|_{\mathcal{U}}^p])^{1/p},$$

and the expected value

$$\mathbf{E}[f] = \int_\Omega f \, \mathrm{d}\mathbf{P}, \quad f \in L_1(\Omega, \mathcal{U}).$$

Filtration: $\{\mathcal{F}_t\}_{t \geq 0} \subseteq \mathcal{F}$ increasing family of sigma algebras, $\mathcal{F}_t \subseteq \mathcal{F}_s$ if $t \leq s$.
Stochastic process: $f = \{f(t)\}_{t \geq 0}$ such that each $f(t)$ is a random variable. It is adapted if $f(t)$ is $\mathcal{F}_t$-measurable.
Note: $f(\omega, t)$, we write $f(t) = f_t = f(\cdot, t)$.

Brownian motion

Probability space $(\Omega, \mathcal{F}, \mathbf{P})$.
Brownian motion: Real-valued stochastic process $\beta = (\beta(t))_{t \geq 0}$ such that

- $\beta(0) = 0$.
- continuous paths $t \mapsto \beta(t)$ for almost every $\omega \in \Omega$.
- independent increments: $\beta(t) - \beta(s)$ is independent of $\beta(r)$ for $0 \leq r \leq s \leq t$.
- Gaussian law: $\mathbf{P} \circ (\beta(t) - \beta(s))^{-1} \sim \mathcal{N}(0, t - s), \quad s \leq t$. In particular, $\mathbf{E}(\beta(t) - \beta(s)) = 0$, $\mathbf{E}(\beta(t) - \beta(s))^2 = t - s$.

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It is a non-trivial fact that Brownian motion exists.

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- Gaussian law: $\mathbf{P} \circ (\beta(t) - \beta(s))^{-1} \sim \mathcal{N}(0, t-s), \quad s \leq t$. In particular, $\mathbf{E}(\beta(t) - \beta(s)) = 0$, $\mathbf{E}(\beta(t) - \beta(s))^2 = t - s$.

It is continuous, but nowhere differentiable. Nevertheless, the Itô integral

$$I = \int_0^T f(t) \, \mathrm{d}\beta(t)$$

can be defined, if the stochastic process f satisfies certain assumptions, some more details later...
It is a random variable: $I(\omega) = (\int_0^T f(t) \, \mathrm{d}\beta(t))(\omega)$. It is not path-wise defined: $I(\omega) \neq \int_0^T f(t,\omega) \, \mathrm{d}\beta(t,\omega)$.

Stochastic ODE

$$\begin{cases} \mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma(X(t), t)\,\mathrm{d}B(t), & t \in [0, T] \\ X(0) = X_0. \end{cases}$$

This means

$$X(t) = X_0 + \int_0^t \mu(X(s), s)\,\mathrm{d}s + \int_0^t \sigma(X(s), s)\,\mathrm{d}B(s), \quad t \in [0, T].$$

Stochastic ODE

$$\begin{cases} dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t), & t \in [0, T] \\ X(0) = X_0. \end{cases}$$

This means

$$X(t) = X_0 + \int_0^t \mu(X(s), s)\, ds + \int_0^t \sigma(X(s), s)\, dB(s), \quad t \in [0, T].$$

Could be a system:

$$dX_i = \mu_i(X_1, \dots, X_n, t)\, dt + \sum_{j=1}^{m} \sigma_{ij}(X_1, \dots, X_n, t)\, d\beta_j(t), \quad i = 1, \dots, n,$$

$X = (X_1, \dots, X_n)^T \in \mathbf{R}^n, \quad \mu : \mathbf{R}^n \times [0, T] \to \mathbf{R}^n, \quad \sigma : \mathbf{R}^n \times [0, T] \to \mathbf{R}^{n \times m}$,

and $B = (\beta_1, \dots, \beta_m)^T$ an m-dimensional Brownian motion, consisting of m independent Brownian motions β_j.
Drift μ, diffusion σ.

Covariance

If σ is a constant matrix:

$\mathrm{d}X(t) = \mu(X(t), t)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t)$

The covariance of the noise term is:
(with increments $\Delta B = B(t + \Delta t) - B(t)$)

$$
\begin{aligned}
\mathbf{E}\big[(\sigma\Delta B) \otimes (\sigma\Delta B)\big] &= \mathbf{E}\big[(\sigma\Delta B)(\sigma\Delta B)^T\big] \\
&= \mathbf{E}\big[\sigma\Delta B\Delta B^T\sigma^T\big] \\
&= \sigma\mathbf{E}\big[\Delta B\Delta B^T\big]\sigma^T \\
&= \sigma\,(\Delta t\, I)\,\sigma^T = \Delta t\,\sigma\sigma^T = \Delta t\, Q
\end{aligned}
$$

Covariance matrix: $Q = \sigma\sigma^T \qquad (n \times m) \times (m \times n) = n \times n$

It is symmetric positive semidefinite.

So $\{\sigma B(t)\}_{t \geq 0}$ is a vector-valued Wiener process with covariance matrix $Q = \sigma\sigma^T$.

Covariance

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&= \sigma\mathbf{E}\left[\Delta B\Delta B^T\right]\sigma^T \\
&= \sigma\,(\Delta t\, I)\,\sigma^T = \Delta t\,\sigma\sigma^T = \Delta t\, Q
\end{aligned}$$

Covariance matrix: $Q = \sigma\sigma^T$ $\qquad (n\times m)\times(m\times n) = n\times n$

It is symmetric positive semidefinite.

So $\{\sigma B(t)\}_{t\geq 0}$ is a vector-valued Wiener process with covariance matrix $Q = \sigma\sigma^T$.

Conversely, given Q we may take $\sigma = Q^{1/2}$ and use $Q^{1/2}\,\mathrm{d}B(t)$.

We want to do this in Hilbert space.

Q-Wiener process

We start with a covariance operator $Q \in \mathcal{L}(\mathcal{U})$, self-adjoint, positive semidefinite. We assume that it has an eigenbasis:

$$Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty} \text{ ON basis in } \mathcal{U}.$$

Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

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Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

Two important cases:

- $\mathrm{Tr}(Q) < \infty$. Then $W(t)$ converges in $L_2(\Omega, \mathcal{U})$:
$$\mathbf{E}\Big\| \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j \Big\|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$

Q-Wiener process

We start with a covariance operator $Q \in \mathcal{L}(\mathcal{U})$, self-adjoint, positive semidefinite. We assume that it has an eigenbasis:

$$Qe_j = \gamma_j e_j, \quad \gamma_j \geq 0, \quad \{e_j\}_{j=1}^{\infty} \text{ ON basis in } \mathcal{U}.$$

Let $\beta_j(t)$ be independent identically distributed, real-valued, Brownian motions. Define

$$W(t) = \sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j.$$

Two important cases:

- $\mathrm{Tr}(Q) < \infty$. Then $W(t)$ converges in $L_2(\Omega, \mathcal{U})$:
  $$\mathbf{E}\Big\|\sum_{j=1}^{\infty} \gamma_j^{1/2} \beta_j(t) e_j\Big\|_{\mathcal{U}}^2 = \sum_{j=1}^{\infty} \gamma_j \mathbf{E}\big(\beta_j(t)^2\big) = t \sum_{j=1}^{\infty} \gamma_j = t\,\mathrm{Tr}(Q) < \infty$$
- $Q = I$, "white noise". Then $W(t)$ is not $\mathcal{U}$-valued, since $\mathrm{Tr}(I) = \infty$, but converges in a weaker sense; i.e., in a larger space $\mathcal{U}_1$.

Q-Wiener process

If $\text{Tr}(Q) < \infty$:

- $W(0) = 0$.
- continuous paths $t \mapsto W(t)$ in $\mathcal{U}$.
- independent increments: $W(t) - W(s)$ is independent of $W(r)$ for $0 \leq r \leq s \leq t$.
- Gaussian law: $\mathbf{P} \circ (W(t) - W(s))^{-1} \sim \mathcal{N}(0, (t-s)Q), \quad s \leq t$

Q-Wiener process

Proof.

(Covariance.) Let $\Delta W = W(t) - W(s)$. Then

$$\begin{aligned}
\big\langle \mathbf{E}\big[\Delta W \otimes \Delta W\big] u, v \big\rangle_{\mathcal{U}} &= \mathbf{E}\big[\langle \Delta W, u\rangle_{\mathcal{U}} \langle \Delta W, v\rangle_{\mathcal{U}}\big] \\
&= \mathbf{E}\Big[\Big\langle \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j e_j, u \Big\rangle_{\mathcal{U}} \Big\langle \sum_{k=1}^{\infty} \gamma_k^{1/2} \Delta\beta_k e_k, v \Big\rangle_{\mathcal{U}}\Big] \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \gamma_j^{1/2} \gamma_k^{1/2} \mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] \langle e_j, u\rangle_{\mathcal{U}} \langle e_k, v\rangle_{\mathcal{U}} \\
&= (t-s) \sum_{j=1}^{\infty} \gamma_j \langle e_j, u\rangle_{\mathcal{U}} \langle e_j, v\rangle_{\mathcal{U}} = (t-s)\langle Qu, v\rangle_{\mathcal{U}},
\end{aligned}$$

because, by independence,

$$\mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] = \begin{cases} \mathbf{E}\big[\Delta\beta_j^2\big] = (t-s), & j = k, \\ \mathbf{E}\big[\Delta\beta_j\big]\, \mathbf{E}\big[\Delta\beta_k\big] = 0, & j \neq k. \end{cases}$$

□

Q-Wiener process

Why Hilbert–Schmidt? Let $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$ and calculate the norm

$$\|B(W(t) - W(s))\|^2_{L_2(\Omega,\mathcal{H})} = \mathbf{E}\big[\|B\Delta W\|^2_{\mathcal{H}}\big]$$

$$= \mathbf{E}\Big[\Big\langle \sum_{j=1}^{\infty} \gamma_j^{1/2} \Delta\beta_j Be_j, \sum_{k=1}^{\infty} \gamma_k^{1/2} \Delta\beta_k Be_k \Big\rangle_{\mathcal{H}}\Big]$$

$$= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \gamma_j^{1/2} \gamma_k^{1/2} \mathbf{E}\big[\Delta\beta_j \Delta\beta_k\big] \langle Be_j, Be_k\rangle_{\mathcal{H}} = (t-s) \sum_{j=1}^{\infty} \gamma_j \|Be_j\|^2_{\mathcal{H}}$$

$$= (t-s) \sum_{j=1}^{\infty} \|B\gamma_j^{1/2} e_j\|^2_{\mathcal{H}} = (t-s) \sum_{j=1}^{\infty} \|BQ^{1/2} e_j\|^2_{\mathcal{H}}$$

$$= (t-s) \|BQ^{1/2}\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = (t-s)\|B\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}.$$

Here we used the Hilbert–Schmidt norm of a linear operator $T \colon \mathcal{U} \to \mathcal{H}$:

$$\|T\|^2_{\mathcal{L}_2(\mathcal{U},\mathcal{H})} = \sum_{j=1}^{\infty} \|T\phi_j\|^2_{\mathcal{H}}, \quad \text{arbitrary ON-basis } \{\phi_j\}_{j=1}^{\infty} \text{ in } \mathcal{U}.$$

Also, it is useful to introduce $\|T\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})} = \|TQ^{1/2}\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}$.

Wiener integral

We want to define $\int_0^T \Phi(t)\,\mathrm{d}W(t)$, where $\Phi \in L_2([0,T], \mathcal{L}_2^0(\mathcal{U},\mathcal{H}))$ is a deterministic integrand. The construction goes in three steps.

1. Simple functions.

$$0 = t_0 < \cdots < t_j < \cdots < t_N = T,\ \Phi = \sum_{j=0}^{N-1} \Phi_j \mathbf{1}_{[t_j,t_{j+1})},\ \Phi_j \in \mathcal{L}_2^0(\mathcal{U},\mathcal{H}).$$

Define

$$\int_0^T \Phi(t)\,\mathrm{d}W(t) = \sum_{j=0}^{N-1} \Phi_j \big(W(t_{j+1}) - W(t_j)\big).$$

Wiener integral

2. Itô isometry for simple functions. Using the independence of increments and the previous norm calculation:

$$\Big\|\int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\|^2_{L_2(\Omega,\mathcal{H})} = \mathbf{E}\Big[\Big\|\sum_{j=0}^{N-1}\Phi_j\big(W(t_{j+1})-W(t_j)\big)\Big\|^2_{\mathcal{H}}\Big]$$

$$= \sum_{j=0}^{N-1}\mathbf{E}\Big[\Big\|\Phi_j\big(W(t_{j+1})-W(t_j)\big)\Big\|^2_{\mathcal{H}}\Big]$$

$$= \sum_{j=0}^{N-1}\|\Phi_j\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}(t_{j+1}-t_j) = \int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}\,\mathrm{d}t.$$

So we have an isometry for simple functions:

$$\Phi \mapsto \int_0^T \Phi\,\mathrm{d}W,$$

$$L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H})) \to L_2(\Omega,\mathcal{H}).$$

3. Extend to all of $L_2([0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))$ by density.

Itô integral

For a random integrand the Itô integral $\int_0^T \Phi \, \mathrm{d}W$ can be defined together with the isometry

$$\mathbf{E}\Big[\Big\|\int_0^T \Phi(t)\,\mathrm{d}W(t)\Big\|_{\mathcal{H}}^2\Big] = \mathbf{E}\Big[\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}\,\mathrm{d}t\Big]$$

or

$$\Big\|\int_0^T \Phi\,\mathrm{d}W\Big\|_{L_2(\Omega,\mathcal{H})} = \|\Phi\|_{L_2(\Omega\times[0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))}$$

Here the process $\Phi\colon [0,T] \to \mathcal{L}_2^0(\mathcal{U},\mathcal{H})$ must be predictable and adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by W and

$$\|\Phi\|^2_{L_2(\Omega\times[0,T],\mathcal{L}_2^0(\mathcal{U},\mathcal{H}))} = \mathbf{E}\Big[\int_0^T \|\Phi(t)\|^2_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})}\,\mathrm{d}t\Big] < \infty.$$

Recall $\|B\|_{\mathcal{L}_2^0(\mathcal{U},\mathcal{H})} = \|BQ^{1/2}\|_{\mathcal{L}_2(\mathcal{U},\mathcal{H})}$.

No details here...

Stochastic evolution equation

$$\mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W, \quad t > 0; \quad X(0) = X_0$$

It is now possible to study the mild form of the above stochastic evolution equation:

$$\begin{aligned} X(t) = E(t)X_0 &+ \int_0^t E(t-s)F(X(s))\,\mathrm{d}s \\ &+ \int_0^t E(t-s)G(X(s))\,\mathrm{d}W(s), \quad t \geq 0 \end{aligned}$$

where $E(t) = \mathrm{e}^{-tA}$ is a C_0-semigroup, see Da Prato and Zabczyk [1].

But here we specialize to the heat and wave equations.

Linear stochastic heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(\xi, t) - \Delta u(\xi, t) = \dot{W}(\xi, t), & \xi \in \mathcal{D} \subseteq \mathbf{R}^d, \ t > 0 \\ u(\xi, t) = 0, & \xi \in \partial\mathcal{D}, \ t > 0 \\ u(\xi, 0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = B\,\mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = H = L_2(\mathcal{D})$, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\mathcal{D} \subseteq \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{U} = H$
- $X(t)$, H-valued stochastic process
- $E(t) = e^{-t\Lambda}$, analytic semigroup generated by $-\Lambda$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

Regularity

$$\|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta \langle v, \phi_j \rangle^2 \Big)^{1/2}, \quad \dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Mean square norm: $\|v\|^2_{L_2(\Omega, \dot{H}^\beta)} = \mathbf{E}(\|v\|^2_{\dot{H}^\beta}), \quad \beta \in \mathbf{R}$

Hilbert–Schmidt norm: $\|T\|_{\mathrm{HS}} = \|T\|_{\mathcal{L}_2(H,H)}$

Theorem. If $\|\Lambda^{(\beta-1)/2}\|_{\mathcal{L}_2^0(H)} = \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then $\|X(t)\|_{L_2(\Omega, \dot{H}^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega, \dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} \Big)$

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Two interesting cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^{\infty} \|Q^{1/2} e_j\|^2 = \sum_{j=1}^{\infty} \gamma_j = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

  This is because $\lambda_j \sim j^{2/d}$, so that $\|\Lambda^{(\beta-1)/2}\|^2_{\mathrm{HS}} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1$, $\beta < 1/2$

Proof with $X_0 = 0$

(Isometry, arbitrary ON-basis $\{\phi_j\}$)

$$
\begin{aligned}
\|X(t)\|^2_{L_2(\Omega,\dot H^\beta)} &= \mathbf{E}\Big(\Big\| \int_0^t \Lambda^{\beta/2} E(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \|\Lambda^{\beta/2} E(s) Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \sum_{k=1}^\infty \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{(\beta-1)/2} Q^{1/2}\phi_k\|^2\,\mathrm{d}s \\
&\le C \sum_{k=1}^\infty \|\Lambda^{(\beta-1)/2} Q^{1/2}\phi_k\|^2 \\
&= C \|\Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}
$$

$$\boxed{\int_0^t \|\Lambda^{1/2} E(s) v\|^2\,\mathrm{d}s \le \tfrac{1}{2}\|v\|^2}$$

Regularity

We used smoothing of order 1 in the form (6):

$$\int_0^t \|\Lambda^{1/2} E(s) v\|^2 \, \mathrm{d}s \leq \tfrac{1}{2} \|v\|^2,$$

which holds for the heat semigroup.
For an analytic semigroup in general we have only (5):

$$\|\Lambda^{\alpha} E(t) v\| \leq C_{\alpha} t^{-\alpha} \|v\|,$$

which leads to

$$\begin{aligned} \|X(t)\|^2_{L_2(\Omega, \dot{H}^{\beta})} &= \sum_{k=1}^{\infty} \int_0^t \|\Lambda^{\frac{1-\epsilon}{2}} E(s) \Lambda^{\frac{\beta+\epsilon-1}{2}} Q^{1/2} \phi_k\|^2 \, \mathrm{d}s \\ &\leq C_{\epsilon} \int_0^t s^{-1+\epsilon} \, \mathrm{d}s \sum_{k=1}^{\infty} \|\Lambda^{\frac{\beta+\epsilon-1}{2}} Q^{1/2} \phi_k\|^2 = C_{\epsilon} \frac{t^{\epsilon}}{\epsilon} \|\Lambda^{\frac{\beta+\epsilon-1}{2}} Q^{1/2}\|^2_{\mathrm{HS}}. \end{aligned}$$

An ϵ-loss!

Temporal regularity for the stochastic heat equation

Take $X_0 = 0$ so that (stochastic convolution) $X(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$.

Theorem

If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,1]$, *then*

$$\|X(t) - X(s)\|_{L_2(\Omega,H)} \leq C|t-s|^{\frac{\beta}{2}} \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}$$

Proof. Take $t > s$ and compute

$$\begin{aligned} X(t) - X(s) &= \int_0^t E(t-r)\,\mathrm{d}W(r) - \int_0^s E(s-r)\,\mathrm{d}W(r) \\ &= \int_0^s \big(E(t-r) - E(s-r)\big)\,\mathrm{d}W(r) \\ &\quad + \int_s^t E(t-r)\,\mathrm{d}W(r) \end{aligned}$$

Temporal regularity

The two terms are independent random variables with zero mean and therefore, using also Itô's isometry,

$$\begin{aligned}\mathbf{E}(\|X(t)-X(s)\|^2) &= \mathbf{E}\Big(\Big\|\int_0^s \big(E(t-s)-I\big)E(s-r)\,\mathrm{d}W(r)\Big\|^2\Big) \\ &\quad + \mathbf{E}\Big(\Big\|\int_s^t E(t-r)\,\mathrm{d}W(r)\Big\|^2\Big) \\ &= \int_0^s \|(E(t-s)-I)E(r)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}r \\ &\quad + \int_0^{t-s} \|E(r)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}r\end{aligned}$$

For the first term we use

$$\begin{aligned}&\|(E(t)-I)v\| \le Ct^{\alpha}\|\Lambda^{\alpha}v\|, \quad t>0,\ \alpha\in[0,1] \\ &\Rightarrow \|(E(t)-I)\Lambda^{-\alpha}v\| \le Ct^{\alpha}\|v\| \\ &\Rightarrow \|(E(t)-I)\Lambda^{-\alpha}\|_{\mathcal{L}(H)} \le Ct^{\alpha}\end{aligned}$$

Temporal regularity

With $\alpha = \beta/2$, $\beta \in [0, 2]$ and using (6) as in the spatial regularity proof:

$$
\begin{aligned}
&\int_0^s \|(E(t-s) - I)E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&\le \int_0^s \|(E(t-s) - I)\Lambda^{-\frac{\beta}{2}}\Lambda^{\frac{\beta}{2}}E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&\le \|(E(t-s) - I)\Lambda^{-\frac{\beta}{2}}\|_{\mathcal{L}(H)}^2 \int_0^s \|\Lambda^{\frac{1}{2}}E(r)\Lambda^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&\le C(t-s)^{\beta}\|\Lambda^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathrm{HS}}^2.
\end{aligned}
$$

For the second term we use a refined version of (6), see [2, Lemma 3.9]:

$$
\int_0^t \|\Lambda^{\frac{\alpha}{2}}E(s)v\|^2 \,\mathrm{d}s \le Ct^{1-\alpha}\|v\|^2, \quad \alpha \in [0, 1],
$$

with $\alpha = 1 - \beta$, $\beta \in [0, 1]$:

$$
\begin{aligned}
&\int_0^{t-s} \|E(r)Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \\
&= \int_0^{t-s} \|\Lambda^{\frac{1-\beta}{2}}E(r)\Lambda^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathrm{HS}}^2 \,\mathrm{d}r \le C(t-s)^{\beta}\|\Lambda^{\frac{\beta-1}{2}}Q^{1/2}\|_{\mathrm{HS}}^2.
\end{aligned}
$$

The linear stochastic wave equation

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$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi, t) - \Delta u(\xi, t) = \dot{W}(\xi, t), & \xi \in \mathcal{D} \subseteq \mathbf{R}^d, \ t > 0 \\ u(\xi, t) = 0, & \xi \in \partial\mathcal{D}, \ t > 0 \\ u(\xi, 0) = u_0, \ \dfrac{\partial u}{\partial t}(\xi, 0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

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$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot{H}^0 = L_2(\mathcal{D})$$

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$$\mathcal{H} = \mathcal{H}^0 = \dot{H}^0 \times \dot{H}^{-1}, \quad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}, \quad D(A) = \mathcal{H}^1$$

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

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  C_0-semigroup on $\mathcal{H}$ but not analytic (actually a group)

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), \quad t > 0 \\ X(0) = X_0 \end{cases}$$

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Here

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})\langle v, \varphi_j\rangle\varphi_j, \quad (\lambda_j, \varphi_j) \text{ are eigenpairs of } \Lambda$$

Regularity

Regularity

Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique mild solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega, \mathcal{H}^\beta)} \leq t \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}, \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}$$

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Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.

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Two cases:

- If $\|Q^{1/2}\|_{\mathrm{HS}}^2 = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ iff $d = 1$, $\beta < 1/2$.

Proof for X_1

Isometry:

$$\begin{aligned}
\|X_1(t)\|^2_{L_2(\Omega,\dot{H}^\beta)} &= \mathbf{E}\Big(\Big\| \int_0^t \Lambda^{\beta/2}\Lambda^{-1/2} \sin\big((t-s)\Lambda^{1/2}\big)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \|\Lambda^{(\beta-1)/2} \sin\big(s\Lambda^{1/2}\big) Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&= \int_0^t \|\sin\big(s\Lambda^{1/2}\big)\Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}\,\mathrm{d}s \\
&\le \int_0^t \|\sin\big(s\Lambda^{1/2}\big)\|^2_{\mathcal{L}(H)}\,\mathrm{d}s\,\|\Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}} \\
&\le t\|\Lambda^{(\beta-1)/2} Q^{1/2}\|^2_{\mathrm{HS}}
\end{aligned}$$

An alternative condition

Note: we do not assume that Λ and Q commute, i.e., we do not assume that they have a common eigenbasis. However, then it may be difficult to verify the condition $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathsf{HS}} < \infty$.
The following theorem gives alternative conditions that imply this.

Theorem

Assume that $Q \in \mathcal{L}(H)$ is selfadjoint, positive semidefinite and that Λ is a densely defined, unbounded, selfadjoint, positive definite, linear operator on H with an orthonormal basis of eigenvectors. Then the following inequalities hold, for $s \in \mathbf{R}$, $\alpha > 0$,

$$\|\Lambda^{\frac{s}{2}}Q^{\frac{1}{2}}\|_{\mathsf{HS}}^2 \leq \|\Lambda^s Q\|_{\mathsf{Tr}} \leq \|\Lambda^{s+\alpha}Q\|_{\mathcal{L}(H)}\|\|\Lambda^{-\alpha}\|_{\mathsf{Tr}}, \tag{8}$$

$$\|\Lambda^{\frac{s}{2}}Q^{\frac{1}{2}}\|_{\mathsf{HS}}^2 \leq \|\Lambda^{s+\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathsf{Tr}}, \tag{9}$$

provided that the respective norms are finite. Furthermore, if Λ and Q have a common basis of eigenvectors, in particular, if $Q = I$, then

$$\|\Lambda^{\frac{s}{2}}Q^{\frac{1}{2}}\|_{\mathsf{HS}}^2 = \|\Lambda^s Q\|_{\mathsf{Tr}} = \|\Lambda^{s+\frac{1}{2}}Q\Lambda^{-\frac{1}{2}}\|_{\mathsf{Tr}}. \tag{10}$$

This is Theorem 2.1 in [4].

An alternative condition

Here $\|T\|_{\mathrm{Tr}} = \|T\|_{\mathcal{L}_1(H)} = \sum_{j=1}^{\infty} \sigma_j$ is the trace norm defined in terms of the singular values σ_j of the trace class operator T, i.e., σ_j are the non-negative square roots of the eigenvalues of TT^*. We have $\|T\|_{\mathrm{Tr}} = \mathrm{Tr}(T)$ if T is self-adjoint positive semidefinite.

Therefore, using (8):

$$\|\Lambda^{\frac{\beta-1}{2}} Q^{\frac{1}{2}}\|_{\mathrm{HS}}^2 \leq \|\Lambda^{\beta-1+\alpha} Q\|_{\mathcal{L}(H)} \|\Lambda^{-\alpha}\|_{\mathrm{Tr}},$$

we select $\alpha > 0$ such that $\mathrm{Tr}(\Lambda^{-\alpha}) = \sum_{j=1}^{\infty} \lambda_j^{-\alpha} < \infty$, which is possible. Then it suffices to verify that $\|\Lambda^{\beta-1+\alpha} Q\|_{\mathcal{L}(H)} < \infty$.

(Schatten classes: $\mathcal{L}_p(H)$, $\|T\|_{\mathcal{L}_p(H)} = (\sum_{j=1}^{\infty} \sigma_j^p)^{1/p}$, $1 \leq p \leq \infty$.)

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Now: proceed to finite elements!

Lecture 2. Strong convergence of finite element approximations

Recall the linear stochastic heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(\xi,t) - \Delta u(\xi,t) = \dot{W}(\xi,t), & \xi \in \mathcal{D} \subseteq \mathbf{R}^d,\ t > 0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t > 0 \\ u(\xi,0) = u_0, & \xi \in \mathcal{D} \end{cases}$$

$$\begin{cases} \mathrm{d}X + AX\,\mathrm{d}t = \mathrm{d}W, & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\mathcal{H} = \mathcal{U} = H = L_2(\mathcal{D})$, $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, $\mathcal{D} \subseteq \mathbf{R}^d$, bounded domain
- $A = \Lambda = -\Delta$, $D(\Lambda) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$, $B = I$
- probability space $(\Omega, \mathcal{F}, \mathbf{P})$
- $W(t)$, Q-Wiener process on $\mathcal{U} = H$
- $X(t)$, $\mathcal{H}$-valued stochastic process
- $E(t) = e^{-t\Lambda}$, analytic semigroup generated by $-\Lambda$

Mild solution (stochastic convolution):

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s), \quad t \geq 0$$

Regularity

$$\|v\|_{\dot{H}^\beta} = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^{\infty} \lambda_j^\beta \langle v, \phi_j \rangle^2 \Big)^{1/2}, \quad \dot{H}^\beta = D(\Lambda^{\beta/2}), \quad \beta \in \mathbf{R}$$

Mean square norm: $\|v\|^2_{L_2(\Omega,\dot{H}^\beta)} = \mathbf{E}(\|v\|^2_{\dot{H}^\beta}), \quad \beta \in \mathbf{R}$

Hilbert–Schmidt norm: $\|T\|_{\mathrm{HS}} = \|T\|_{\mathcal{L}_2(H,H)}$

Theorem. If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then $\|X(t)\|_{L_2(\Omega,\dot{H}^\beta)} \leq C\Big(\|X_0\|_{L_2(\Omega,\dot{H}^\beta)} + \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}\Big)$

Two cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \sum_{j=1}^{\infty} \|Q^{1/2} e_j\|^2 = \sum_{j=1}^{\infty} \gamma_j = \mathrm{Tr}(Q) < \infty$, then $\beta = 1$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial \xi^2}$, then $\|\Lambda^{(\beta-1)/2}\|_{\mathrm{HS}} < \infty$ for $\beta < 1/2$.

  $\|\Lambda^{(\beta-1)/2}\|^2_{\mathrm{HS}} = \sum_j \lambda_j^{-(1-\beta)} \approx \sum_j j^{-(1-\beta)2/d} < \infty$ iff $d = 1$, $\beta < 1/2$

The finite element method

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}, \quad S_h \subseteq H_0^1(\mathcal{D}) = \dot{H}^1$
- S_h continuous piecewise linear functions
- $X_h(t) \in S_h;\ \langle \mathrm{d}X_h, \chi\rangle + \langle \nabla X_h, \nabla \chi\rangle \,\mathrm{d}t = \langle \mathrm{d}W, \chi\rangle\ \forall \chi \in S_h,\ t > 0$
- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $\langle \Lambda_h \psi, \chi\rangle = \langle \nabla \psi, \nabla \chi\rangle\ \forall \psi, \chi \in S_h$
- $A_h = \Lambda_h$
- $P_h \colon L_2 \to S_h,$ orthogonal projection, $\quad \langle P_h f, \chi\rangle = \langle f, \chi\rangle\ \forall \chi \in S_h$

$$\begin{cases} X_h(t) \in S_h, \quad X_h(0) = P_h X_0 \\ \mathrm{d}X_h + \Lambda_h X_h \,\mathrm{d}t = P_h \,\mathrm{d}W, \quad t > 0 \end{cases}$$

$P_h W(t)$ is a Q_h-Wiener process with $Q_h = P_h Q P_h$.

Mild solution, with $E_h(t) v_h = \mathrm{e}^{-t\Lambda_h} v_h = \sum_{j=1}^{N_h} \mathrm{e}^{-t\lambda_{h,j}} \langle v_h, \varphi_{h,j}\rangle \varphi_{h,j}$:

$$X_h(t) = E_h(t) P_h X_0 + \int_0^t E_h(t-s) P_h \,\mathrm{d}W(s)$$

Error analysis for elliptic problems

$u \in H_0^1: \quad \langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in H_0^1 \qquad$ or $\Lambda u = f$

$u_h \in S_h: \quad \langle \nabla u_h, \nabla \chi \rangle = \langle f, \chi \rangle \quad \forall \chi \in S_h \qquad$ or $\Lambda_h u_h = P_h f$

Then $u_h = R_h u$, where R_h is the Ritz projector:

$R_h \colon H_0^1 \to S_h$
$\langle \nabla R_h u, \nabla \chi \rangle = \langle \nabla u, \nabla \chi \rangle \quad \forall \chi \in S_h$

Error estimate (using elliptic regularity, Aubin-Nitsche duality argument):

$\|R_h u - u\| \leq Ch^2 \|u\|_{H^2} \quad \forall u \in H^2 \cap H_0^1$

But $\dot{H}^2 = H^2 \cap H_0^1$ with equivalent norms, so that

$\|R_h u - u\| \leq Ch^2 \|u\|_{\dot{H}^2} \quad \forall u \in \dot{H}^2$

But $\|u\|_{\dot{H}^2} = \|\Lambda u\| = \|f\|$, so that

$\|\Lambda_h^{-1} P_h f - \Lambda^{-1} f\| \leq Ch^2 \|f\| \quad \forall f \in H$

or

$\|\Lambda_h^{-1} P_h - \Lambda^{-1}\|_{\mathcal{L}(H)} \leq Ch^2$

Also: $\quad \|P_h v - v\| \leq Ch^2 \|v\|_{\dot{H}^2} \quad \forall v \in \dot{H}^2$

Approximation of the semigroup

$$\begin{cases} u_t + \Lambda u = 0, & t > 0 \\ u(0) = v \end{cases} \qquad \begin{cases} u_{h,t} + \Lambda_h u_h = 0, & t > 0 \\ u_h(0) = P_h v \end{cases}$$

$$u(t) = E(t)v \qquad\qquad u_h(t) = E_h(t)P_h v$$

Denote

$$F_h(t)v = E_h(t)P_h v - E(t)v, \quad \|v\|_\beta = \|\Lambda^{\beta/2} v\|.$$

We have, for $0 \le \beta \le 2$,

- $\|F_h(t)v\| \le Ch^\beta \|v\|_\beta, \quad t \ge 0$
- $\Big(\int_0^t \|F_h(s)v\|^2 \,\mathrm{d}s \Big)^{1/2} \le Ch^\beta \|v\|_{\beta-1}, \quad t \ge 0$
- $\|F_h(t)v\| \le Ch^\beta t^{-\beta/2} \|v\|, \quad t > 0$

First prove for $\beta = 2$ and $\beta = 0$, then interpolate. This can be found in Thomée [4, Chapters 1, 3]. Note: $\beta = 2$ is the maximal order for piecewise linear finite elements.
(FEM of order $r \ge 2$ are piecewise poly of degree $r - 1 \ge 1$.)

Proof the second error estimate

We prove the error estimate for $\beta = 2$ and $\beta = 0$, we then interpolate to $\beta \in [0, 2]$.

For $\beta = 2$ we must prove $\Big(\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s \Big)^{1/2} \le Ch^2 \|v\|_1$.

Recall $F_h(t)v = E_h(t)P_h v - E(t)v = u_h(t) - u(t)$, where

$$\begin{aligned} \langle u_t, \phi \rangle + \langle \nabla u, \nabla \phi \rangle &= 0 \quad \forall \phi \in H_0^1(\mathcal{D}); \quad u(0) = v \\ \langle u_{h,t}, \phi_h \rangle + \langle \nabla u_h, \nabla \phi_h \rangle &= 0 \quad \forall \phi_h \in S_h; \quad u_h(0) = P_h v \end{aligned}$$

Take $\phi = \phi_h$ and subtract (with $e = u_h - u$):

$$\langle e_t, \phi_h \rangle + \langle \nabla e, \nabla \phi_h \rangle = 0 \quad \forall \phi_h \in S_h$$

Write $e = (u_h - P_h u) + (P_h u - u) = \theta + \rho$. Then $\theta(0) = 0$ and

$$\langle \theta_t, \phi_h \rangle + \langle \nabla \theta, \nabla \phi_h \rangle = -\langle \rho_t, \phi_h \rangle - \langle \nabla \rho, \nabla \phi_h \rangle = -\langle \nabla \rho, \nabla \phi_h \rangle \quad \forall \phi_h \in S_h$$

Proof

We have

$$\langle \theta_t, \phi_h \rangle + \langle \nabla \theta, \nabla \phi_h \rangle = -\langle \nabla \rho, \nabla \phi_h \rangle \quad \forall \phi_h \in S_h; \qquad \theta(0) = 0.$$

Take $\phi_h = \Lambda_h^{-1}\theta$:

$$\langle \theta_t, \Lambda_h^{-1}\theta \rangle + \langle \nabla \theta, \nabla \Lambda_h^{-1}\theta \rangle = -\langle \nabla \rho, \nabla \Lambda_h^{-1}\theta \rangle$$

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda_h^{-1/2}\theta\|^2 + \|\theta\|^2 = -\langle \rho, \theta \rangle \le \|\rho\|\,\|\theta\| \le \tfrac{1}{2}\|\rho\|^2 + \tfrac{1}{2}\|\theta\|^2$$

$$\int_0^t \|\theta\|^2 \,\mathrm{d}s \le \int_0^t \|\rho\|^2 \,\mathrm{d}s$$

Finally, by $e = \theta + \rho$ and smoothing of order 1, see (6),

$$\int_0^t \|e\|^2 \,\mathrm{d}s \le 2\int_0^t \|\rho\|^2 \,\mathrm{d}s \le Ch^4 \int_0^t \|u\|_2^2 \,\mathrm{d}s \le Ch^4 \|v\|_1^2.$$

Proof

For $\beta = 0$, smoothing of order 1, see (6), holds analogously for E_h:

$$\int_0^t \|\Lambda_h^{1/2} E_h(s) v_h\|^2 \, \mathrm{d}s \leq \tfrac{1}{2} \|v_h\|^2$$

Then

$$\begin{aligned}
\int_0^t \|F_h(s)v\|^2 \, \mathrm{d}s &\leq 2 \int_0^t \|E_h(s) P_h v\|^2 \, \mathrm{d}s + 2 \int_0^t \|E(s)v\|^2 \, \mathrm{d}s \\
&= 2 \int_0^t \|\Lambda_h^{1/2} E_h(s) \Lambda_h^{-1/2} P_h v\|^2 \, \mathrm{d}s \\
&\quad + 2 \int_0^t \|\Lambda^{1/2} E(s) \Lambda^{-1/2} v\|^2 \, \mathrm{d}s \\
&\leq \|\Lambda_h^{-1/2} P_h v\|^2 + \|\Lambda^{-1/2} v\|^2 \leq 2\|v\|_{-1}^2
\end{aligned}$$

Proof of $\|\Lambda_h^{-1/2} P_h v\| \leq \|\Lambda^{-1/2} v\| = \|v\|_{-1}$ on the next page.

Proof

Proof of $\|\Lambda_h^{-1/2} P_h v\| \leq \|\Lambda^{-1/2} v\| = \|v\|_{-1}$.

$$
\begin{aligned}
\|\Lambda_h^{-\frac{1}{2}} P_h v\| &= \sup_{v_h \in S_h} \frac{|(\Lambda_h^{-\frac{1}{2}} P_h v, v_h)|}{\|v_h\|} = \sup_{v_h \in S_h} \frac{|(v, \Lambda_h^{-\frac{1}{2}} v_h)|}{\|v_h\|} \\
&= \sup_{w_h \in S_h} \frac{|(v, w_h)|}{\|\Lambda_h^{\frac{1}{2}} w_h\|} = \sup_{w_h \in S_h} \frac{|(v, w_h)|}{\|w_h\|_1} \\
&\leq \sup_{w \in \dot{H}^1} \frac{|(v, w)|}{\|w\|_1} = \sup_{h \in H} \frac{|(v, \Lambda^{-\frac{1}{2}} h)|}{\|h\|} \\
&= \sup_{h \in H} \frac{|(\Lambda^{-\frac{1}{2}} v, h)|}{\|h\|} = \|\Lambda^{-\frac{1}{2}} v\| = \|v\|_{-1}.
\end{aligned}
$$

Strong convergence

Theorem

If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$*, then*

$$\|X_h(t) - X(t)\|_{L_2(\Omega,H)} \leq Ch^{\beta}\Big(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big).$$

Optimal result: the order of regularity equals the order of convergence.

Two cases:

- If $\|Q^{1/2}\|^2_{\mathrm{HS}} = \mathrm{Tr}(Q) < \infty$, then the convergence rate is $O(h)$.
- If $Q = I,\ d = 1,\ \Lambda = -\frac{\partial^2}{\partial\xi^2}$, then the rate is almost $O(h^{1/2})$.

No result for $Q = I$, $d \geq 2$.

Strong convergence: proof

$$X(t) = E(t)X_0 + \int_0^t E(t-s)\,\mathrm{d}W(s)$$

$$X_h(t) = E_h(t)P_hX_0 + \int_0^t E_h(t-s)\,P_h\,\mathrm{d}W(s)$$

$$F_h(t) = E_h(t)P_h - E(t)$$

$$X_h(t) - X(t) = F_h(t)X_0 + \int_0^t F_h(t-s)\,\mathrm{d}W(s) = e_1(t) + e_2(t)$$

$$\|F_h(t)X_0\| \leq Ch^\beta \|X_0\|_\beta$$

$$\Rightarrow \quad \|e_1(t)\|_{L_2(\Omega,H)} \leq Ch^\beta \|X_0\|_{L_2(\Omega,\dot{H}^\beta)}$$

Strong convergence: proof

$$\begin{cases} \mathbf{E}\Big\| \int_0^t B(s)\,\mathrm{d}W(s)\Big\|^2 = \mathbf{E}\int_0^t \|B(s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s \text{ (isometry)} \\ \Big(\int_0^t \|F_h(s)v\|^2\,\mathrm{d}s\Big)^{1/2} \le Ch^\beta \|v\|_{\beta-1}, \text{ with } v = Q^{1/2}\varphi_j \end{cases}$$

$\Rightarrow$

$$\begin{aligned} \|e_2(t)\|_{L_2(\Omega,H)}^2 &= \mathbf{E}\Big\| \int_0^t F_h(t-s)\,\mathrm{d}W(s)\Big\|^2 = \int_0^t \|F_h(t-s)Q^{1/2}\|_{\mathrm{HS}}^2\,\mathrm{d}s \\ &= \sum_{l=1}^{\infty} \int_0^t \|F_h(t-s)Q^{1/2}\varphi_j\|^2\,\mathrm{d}s \le C\sum_{l=1}^{\infty} h^{2\beta}\|Q^{1/2}\varphi_j\|_{\beta-1}^2 \\ &= Ch^{2\beta}\sum_{l=1}^{\infty}\|\Lambda^{(\beta-1)/2}Q^{1/2}\varphi_j\|^2 = Ch^{2\beta}\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}^2 \end{aligned}$$

Another type of estimate

Take $X_0 = 0$ so that $X(t) = W_\Lambda(t) = \int_0^t E(t-s)\,\mathrm{d}W(s)$.

We have shown

$$\Big(\sup_{t\in[0,T]} \mathbf{E}\Big[\|W_\Lambda(t)\|_\beta^2\Big]\Big)^{1/2} \le C\|\Lambda^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathrm{HS}}$$

$$\Big(\sup_{t\in[0,T]} \mathbf{E}\Big[\|W_{\Lambda_h}(t) - W_\Lambda(t)\|^2\Big]\Big)^{1/2} \le Ch^\beta\|\Lambda^{\frac{\beta-1}{2}} Q^{1/2}\|_{\mathrm{HS}}$$

Theorem

Let $\epsilon \in (0,1]$ and $p > 2/\epsilon$. Then

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]} \|W_\Lambda(t)\|_\beta^p\Big]\Big)^{1/p} \le C_\epsilon\|\Lambda^{\frac{\beta-1}{2}+\epsilon} Q^{1/2}\|_{\mathrm{HS}}$$

$$\Big(\mathbf{E}\Big[\sup_{t\in[0,T]} \|W_{\Lambda_h}(t) - W_\Lambda(t)\|^p\Big]\Big)^{1/p} \le C_\epsilon h^\beta\|\Lambda^{\frac{\beta-1}{2}+\epsilon} Q^{1/2}\|_{\mathrm{HS}}$$

Another type of estimate; proof

We present the idea of the regularity estimate. The proof is an adaptation of the 'factorization method' in the proof of [1, Theorem 5.9, Remark 5.11]: $\quad (E(t-s) \neq E(t)E(-s))$

$$
\begin{aligned}
W_\Lambda(t) &= \int_0^t E(t-\sigma)\,\mathrm{d}W(\sigma) \\
&= c_\alpha \int_0^t E(t-\sigma) \int_\sigma^t (t-s)^{-1+\alpha}(s-\sigma)^{-\alpha}\,\mathrm{d}s\,\mathrm{d}W(\sigma) \\
&= c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma)\,\mathrm{d}s \\
&= c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) Y(s)\,\mathrm{d}s \\
Y(s) &= \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma) \\
c_\alpha^{-1} &= \int_\sigma^t (t-s)^{-1+\alpha}(s-\sigma)^{-\alpha}\,\mathrm{d}s = \frac{\pi}{\sin(\pi\alpha)}
\end{aligned}
$$

Another type of estimate; proof

Idea of the proof:

$$Y(s) = \int_0^s (s-\sigma)^{-\alpha} E(s-\sigma)\,\mathrm{d}W(\sigma)$$

$$W_\Lambda(t) = c_\alpha \int_0^t (t-s)^{-1+\alpha} E(t-s) Y(s)\,\mathrm{d}s$$

Hölder:

$$\|\Lambda^{\frac{\beta}{2}} W_\Lambda(t)\|^p \leq c_\alpha \Big(\int_0^T \big(s^{-1+\alpha}\|E(s)\|\big)^{\frac{p}{p-1}}\,\mathrm{d}s \Big)^{p-1} \int_0^T \|\Lambda^{\frac{\beta}{2}} Y(s)\|^p\,\mathrm{d}s$$

and, hence,

$$\mathbf{E}\Big[\sup_{t\in[0,T]} \|\Lambda^{\frac{\beta}{2}} W_\Lambda(t)\|^p\Big] \leq c_\alpha \Big(\int_0^T \big(s^{-1+\alpha}\|E(s)\|\big)^{\frac{p}{p-1}}\,\mathrm{d}s \Big)^{p-1} \times \int_0^T \mathbf{E}\big[\|\Lambda^{\frac{\beta}{2}} Y(s)\|^p\big]\,\mathrm{d}s$$

Time discretization

$$\begin{cases} dX + AX\,dt = dW, & t > 0 \\ X(0) = X_0 \end{cases}$$

The implicit Euler method:
$k = \Delta t,\ t_n = nk,\ \Delta W^n = W(t_n) - W(t_{n-1})$

$$\begin{cases} X_h^n \in S_h, \quad X_h^0 = P_h X_0 \\ X_h^n - X_h^{n-1} + kA_h X_h^n = P_h \Delta W^n, \end{cases}$$

$$X_h^n = E_{kh} X_h^{n-1} + E_{kh} P_h \Delta W^n, \quad E_{kh} = (I + kA_h)^{-1}$$

$$X_h^n = E_{kh}^n P_h X_0 + \sum_{j=1}^{n} E_{kh}^{n-j+1} P_h \Delta W^j$$

$$X(t_n) = E(t_n) X_0 + \int_0^{t_n} E(t_n - s)\,dW(s)$$

Approximation of the semigroup

Denote $F_n = E_{kh}^n P_h - E(t_n)$

We have the following estimates for $0 \leq \beta \leq 2$:

- $\|F_n v\| \leq C(k^{\beta/2} + h^{\beta})\|v\|_{\beta}$
- $\Big(k \sum_{j=1}^{n} \|F_j v\|^2\Big)^{1/2} \leq C(k^{\beta/2} + h^{\beta})\|v\|_{\beta-1}$

See Thomée [4].

Strong convergence

Theorem

If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ *for some* $\beta \in [0,2]$*, then, with* $e^n = X_h^n - X(t_n)$*,*

$$\|e^n\|_{L_2(\Omega,H)} \leq C(k^{\beta/2} + h^{\beta})\Big(\|X_0\|_{L_2(\Omega,\dot{H}^{\beta})} + \|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}}\Big)$$

The reason why we can have k^1 (when $\beta = 2$) is that the Euler-Maruyama method is exact in the stochastic integral for additive noise. For multiplicative noise we get at most $k^{1/2}$.

J. Printems [3] (only time-discretization)

Y. Yan [1, 2]

Recall the linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(\xi,t) - \Delta u(\xi,t) = \dot W(\xi,t), & \xi \in \mathcal{D} \subseteq \mathbf{R}^d,\ t>0 \\ u(\xi,t) = 0, & \xi \in \partial\mathcal{D},\ t>0 \\ u(\xi,0) = u_0,\ \dfrac{\partial u}{\partial t}(\xi,0) = u_1, & \xi \in \mathcal{D} \end{cases}$$

$$\Lambda = -\Delta, \quad D(\Lambda) = \dot H^2 = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$$

$$\dot H^\beta = D(\Lambda^{\beta/2}), \quad \|v\|_\beta = \|\Lambda^{\beta/2} v\| = \Big(\sum_{j=1}^\infty \lambda_j^\beta \langle v, \varphi_j \rangle^2 \Big)^{1/2}, \quad \beta \in \mathbf{R}$$

$$\begin{bmatrix} \mathrm{d}u \\ \mathrm{d}u_t \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix} \begin{bmatrix} u \\ u_t \end{bmatrix} \mathrm{d}t = \begin{bmatrix} 0 \\ I \end{bmatrix} \mathrm{d}W,$$

$$X = \begin{bmatrix} u \\ u_t \end{bmatrix},\ A = \begin{bmatrix} 0 & -I \\ \Lambda & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad \mathcal{U} = \dot H^0 = L_2(\mathcal{D})$$

$$\mathcal{H}^\beta = \dot H^\beta \times \dot H^{\beta-1}, \quad \mathcal{H} = \mathcal{H}^0 = \dot H^0 \times \dot H^{-1}, \quad D(A) = \mathcal{H}^1$$

Abstract framework

$$\begin{cases} \mathrm{d}X(t) + AX(t)\,\mathrm{d}t = B\,\mathrm{d}W(t), & t > 0 \\ X(0) = X_0 \end{cases}$$

- $\{X(t)\}_{t\geq 0}$, $\mathcal{H} = \dot{H}^0 \times \dot{H}^{-1}$-valued stochastic process
- $\{W(t)\}_{t\geq 0}$, $\mathcal{U} = \dot{H}^0$-valued Q-Wiener process w.r.t. $\{\mathcal{F}_t\}_{t\geq 0}$
- $E(t) = e^{-tA} = \begin{bmatrix} \cos(t\Lambda^{1/2}) & \Lambda^{-1/2}\sin(t\Lambda^{1/2}) \\ -\Lambda^{1/2}\sin(t\Lambda^{1/2}) & \cos(t\Lambda^{1/2}) \end{bmatrix}$,
  C_0-semigroup on $\mathcal{H}$ (actually a group) but not analytic

Here

$$\cos(t\Lambda^{1/2})v = \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})\langle v, \varphi_j\rangle \phi_j, \quad (\lambda_j, \phi_j) \text{ are eigenpairs of } \Lambda$$

Regularity

Theorem. (With $X(0) = 0$ for simplicity.) If $\|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \geq 0$, then there exists a unique mild solution

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \int_0^t E(t-s) B \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda^{-1/2} \sin\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda^{1/2}\right) \mathrm{d}W(s) \end{bmatrix}$$

and

$$\|X(t)\|_{L_2(\Omega, \mathcal{H}^\beta)} \leq C(t) \|\Lambda^{(\beta-1)/2} Q^{1/2}\|_{\mathrm{HS}}, \qquad \mathcal{H}^\beta = \dot{H}^\beta \times \dot{H}^{\beta-1}.$$

Spatial discretization

- triangulations $\{\mathcal{T}_h\}_{0<h<1}$, mesh size h
- finite element spaces $\{S_h\}_{0<h<1}$
- $S_h \subseteq \dot{H}^1 = H_0^1(\mathcal{D})$ continuous piecewise polynomials of degree ≤ 1
- $\Lambda_h \colon S_h \to S_h,$ discrete Laplacian, $\quad \langle \Lambda_h \psi, \chi \rangle = \langle \nabla \psi, \nabla \chi \rangle, \ \forall \chi \in S_h$
- $P_h \colon \dot{H}^0 \to S_h,$ orthogonal projection, $\quad \langle P_h f, \chi \rangle = \langle f, \chi \rangle, \ \forall \chi \in S_h$
- $A_h = \begin{bmatrix} 0 & -I \\ \Lambda_h & 0 \end{bmatrix}, \quad B_h = \begin{bmatrix} 0 \\ P_h \end{bmatrix}$
- $\begin{cases} \mathrm{d}X_h(t) + A_h X_h(t)\,\mathrm{d}t = B_h\,\mathrm{d}W(t), \quad t > 0 \\ X_h(0) = X_{0,h} \end{cases}$
- $E_h(t) = e^{-tA_h} = \begin{bmatrix} \cos(t\Lambda_h^{1/2}) & \Lambda_h^{-1/2}\sin(t\Lambda_h^{1/2}) \\ -\Lambda_h^{1/2}\sin(t\Lambda_h^{1/2}) & \cos(t\Lambda_h^{1/2}) \end{bmatrix}$

Spatial discretization

The mild solution is:

$$X_h(t) = \begin{bmatrix} X_{h,1}(t) \\ X_{h,2}(t) \end{bmatrix}$$

$$= \int_0^t E_h(t-s) B_h \, \mathrm{d}W(s) = \begin{bmatrix} \int_0^t \Lambda_h^{-1/2} \sin\left((t-s)\Lambda_h^{1/2}\right) P_h \, \mathrm{d}W(s) \\ \int_0^t \cos\left((t-s)\Lambda_h^{1/2}\right) P_h \, \mathrm{d}W(s) \end{bmatrix}$$

where, for example,

$$\cos(t\Lambda_h^{1/2})v = \sum_{j=1}^{N_h} \cos\left(t\sqrt{\lambda_{h,j}}\right) \langle v, \varphi_{h,j} \rangle \varphi_{h,j},$$

and $(\lambda_{h,j}, \varphi_{h,j})$ are eigenpairs of Λ_h.

Spatially semidiscrete: approximation of the semigroup

$$\begin{cases} v_{tt}(t) + \Lambda v(t) = 0, \ t > 0 \\ v(0) = 0, \ v_t(0) = f \end{cases} \Rightarrow v(t) = \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right) f$$

$$\begin{cases} v_{h,tt}(t) + \Lambda_h v_h(t) = 0, \ t > 0 \\ v_h(0) = 0, \ v_{h,t}(0) = P_h f \end{cases} \Rightarrow v_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h f$$

We have, for $K_h(t) = \Lambda_h^{-1/2} \sin\left(t\Lambda_h^{1/2}\right) P_h - \Lambda^{-1/2} \sin\left(t\Lambda^{1/2}\right)$ and $r = 2$,

$\|K_h(t)f\| \leq C(t)h^2 \|f\|_{\dot{H}^2}$ "initial regularity of order 3"

$\|K_h(t)f\| \leq 2\|f\|_{\dot{H}^{-1}}$ "initial regularity of order 0" (stability)

$\|K_h(t)f\| \leq C(t)h^{2\frac{\beta}{3}} \|f\|_{\dot{H}^{\beta-1}}, \quad 0 \leq \beta \leq 3$

$\beta - 1$ can not be replaced by $\beta - 1 - \epsilon$ for $\epsilon > 0$ (J. Rauch 1985)

Note: $\|v(t)\|_{\dot{H}^2} \leq \|f\|_{\dot{H}^1}$ "initial regularity of order 2"

Spatially semidiscrete: Strong convergence

Theorem. If $\|\Lambda^{(\beta-1)/2}Q^{1/2}\|_{\mathrm{HS}} < \infty$ for some $\beta \in [0, 3]$, then

$$\|X_{h,1}(t) - X_1(t)\|_{L_2(\Omega,\dot{H}^0)} \le C(t)\, h^{2\frac{\beta}{3}} \big\|\Lambda^{(\beta-1)/2}Q^{1/2}\big\|_{\mathrm{HS}}$$

Higher order FEM: $\quad O(h^{r\frac{\beta}{r+1}}), \quad \beta \in [0, r+1]$.
FEM of order $r \ge 2$ are piecewise poly of degree $r - 1 \ge 1$.

Proof. $\{f_k\}$ an arbitrary ON basis in $\dot{H}^0$

$$\begin{aligned}
\|X_{h,1}(t) - X_1(t)\|^2_{L_2(\Omega,\dot{H}^0)} &= \mathbf{E}\big(\|X_{h,1}(t) - X_1(t)\|^2\big) \\
&= \mathbf{E}\Big(\Big\|\int_0^t K_h(t-s)\,\mathrm{d}W(s)\Big\|^2\Big) \\
&= \int_0^t \big\|K_h(s)Q^{1/2}\big\|^2_{\mathrm{HS}}\,\mathrm{d}s = \int_0^t \sum_{k=1}^{\infty} \big\|K_h(s)Q^{1/2}f_k\big\|^2\,\mathrm{d}s \\
&\le C(t)h^{4\frac{\beta}{3}} \sum_{k=1}^{\infty} \big\|Q^{1/2}f_k\big\|^2_{\dot{H}^{\beta-1}} = C(t)h^{4\frac{\beta}{3}} \big\|\Lambda^{(\beta-1)/2}Q^{1/2}\big\|^2_{\mathrm{HS}}
\end{aligned}$$

This is from [1].
Time stepping is studied in [3].

Nonlinear problems

This kind of analysis carries over (with some limitations) to nonlinear problems

$$\mathrm{d}X + AX\,\mathrm{d}t = F(X)\,\mathrm{d}t + G(X)\,\mathrm{d}W$$

if the operators F, G are globally Lipschitz in the appropriate senses.

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if the operators F, G are globally Lipschitz in the appropriate senses.

For example:

$$\begin{aligned}
&\{E(t)\}_{t\geq 0} \text{ analytic}, \quad \mathrm{Tr}(Q) < \infty \\
&\|F(u) - F(v)\|_H \leq C\|u - v\|_H \\
&\|(G(u) - G(v))Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C\|u - v\|_H
\end{aligned}$$

Then we have spatial regularity: $L_p(\Omega, \dot{H}^\gamma)$, and temporal regularity: Hölder $\gamma/2$ in $L_p(\Omega, H)$ for $\gamma \in [0,1)$, $p \geq 2$, see [1].

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Jentzen and Röckner [2] introduced a linear growth bound:

$$\|A^{\frac{\beta-1}{2}} G(u)Q^{\frac{1}{2}}\|_{\mathcal{L}_2(\mathcal{U},H)} \leq C(1 + \|u\|_{\dot{H}^{\beta-1}})$$

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Kruse and L [2] assumed that A is self-adjoint with compact inverse so that the "special smoothing of order 1" (6) holds.

Then, for $\beta \in [0, 2)$, $p \in [1, \infty)$,

$$\|X(t)\|_{L_p(\Omega,\dot{H}^\beta)} \leq C,$$

and Hölder in t with exponent $\min(\frac{1}{2}, \frac{\beta}{2})$ in $L_p(\Omega, H)$.

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